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Contents

1 Commutative Algebra

1.1 Basics

- Let $A \to B$ ring map, M, N be B-modules. We have a coequalizer diagram:

$$M \otimes_A B \otimes_A N \rightrightarrows M \otimes_A N \longrightarrow M \otimes_B N$$

  This gives another interpretation for tensor products of modules over an A-algebra.

- (Nakayama). Let M f.g. R-module, if $J(R)M = M$, then $M = 0$.
- (In proving Nakayama): for a matrix B, there exists $\operatorname{Adj} B$ such that $\operatorname{Adj} B \cdot B = \det B \cdot I$.
- If there is a prime factorization for an element in the ring, then it is unique. The same holds for prime ideal factorizations. The converse is true in UFDs (former) or Dedekind domains (latter).
  - Geometrically, such decomposition gives a decomposition of $V(I)$ to irreducible components. Such a decomposition corresponds to $I = \bigcap \mathfrak{p}_i$ but not $I = \prod \mathfrak{p}_i$. It corresponds to a prime ideal factorization if any pair of prime ideals are coprime (which is true in a Dedekind domain).
- An ideal $I \subset A$ is primary if the zero-divisors A/I are all nilpotent. I primary implies that $\sqrt{I}$ is prime.
- One can show that prime ideals and primary ideals in $S^{-1}A$ correspond to those in A that do not intersect S. Also, one can show that all ideals in $S^{-1}A$ come from extension.
- We can define the maximal $\tilde{S}$ multiplicative closed, such that $\tilde{S}^{-1}A = S^{-1}A$: we define $\tilde{S} = \{a \in A, ab \in S \text{ for some } b\} = \{a \in A, \frac{a}{1} \text{ unit in } S^{-1}A\}$.
- The residue field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = A/\mathfrak{p}_{A/\mathfrak{p}\setminus 0} = QF(A/\mathfrak{p})$, which follows because localization commutes with taking quotients.
- Localization at a larger set "overloads" the localization at a smaller set. For example, $S \subset T$ implies $(A_S)_T = A_T$. In particular this applies to localization at prime ideals.
- $\operatorname{Supp} M = V(\operatorname{ann} M)$ in $\operatorname{Spec} A$ if M is finitely generated:
  - Localization commutes with finite intersections but not intersections in general; in particular, $(\operatorname{ann} M)_\mathfrak{p} = (\bigcap \operatorname{ann} m_i)_\mathfrak{p} = \bigcap(\operatorname{ann} m_i)_\mathfrak{p} = \bigcap(\operatorname{ann}(m_i)_\mathfrak{p}) = \operatorname{ann} M_\mathfrak{p}$.
  - Therefore, $M_\mathfrak{p} = 0$ iff $\operatorname{ann} M_\mathfrak{p} = A_\mathfrak{p}$, iff $(\operatorname{ann} M)_\mathfrak{p} = A_\mathfrak{p}$, iff $\operatorname{ann} M \not\subset \mathfrak{p}$.
  - This is false in general. For example, consider $A = k[x_1, x_2, ...]$, $M = \bigoplus A/(x_i)$, then $M_{(0)} = 0$, although $\operatorname{ann} M = 0$ as well.
  - In particular, if $x \neq 0$ in $A_\mathfrak{p}$, then $\operatorname{ann} x \subset \mathfrak{p}$.

- Recall that a f.g. A-module can be viewed as a quasi-coherent sheaf that is locally of finite type over $\operatorname{Spec} A$, and it is zero iff every fiber is zero.
  - Let $f : A \to B$ ring map. M finite B-module. If $M \otimes \kappa(\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} f(\operatorname{Spec} B)$, then $M = 0$.
  - $M \otimes \kappa(\mathfrak{p})$ depicts the fiber over $\mathfrak{p}$ of the pushforward of M. Geometrically, the locally finite type quasi-coherent sheaf itself is zero iff every fiber of the pushforward is zero.
  - In general this does not holds for arbitrary quasi-coherent sheaf. Also in general this does not hold if we restrict to only the maximal ideals $\mathfrak{p}$.
- Maximal ideals in $k[x_1, ..., x_n]$ are in the form of $\mathfrak{m} \cap k[x_1, ..., x_n]$, where $\mathfrak{m}$ is maximal in $\bar{k}[x_1, ..., x_n]$. In general, if $A \to B$ is integral, then the induced map on spectrum is surjective for maximal ideals.
  - For example, maximal ideals in $\mathbb{R}[x, y]$ are in the form $(x-a, y-b)$ or $(x-a, y^2+by+c)$ where each quadratic is irreducible over $\mathbb{R}$.
  - Note that for example (x^2+1, y^2+1) is not maximal, since $\mathbb{R}[x, y]/(x^2+1, y^2+1) = \mathbb{C}[y]/(y^2+1) = \mathbb{C} \times \mathbb{C}$ is not a field.
- An element algebraic over $k = QF(A)$ does not imply it is integral over A. For example, in $k[x, x^{-1}]$, x^{-1} is algebraic over $k(x)$ but is not integral over $k[x]$.

1.2 Flatness

- Flatness is detected locally: if M is a B-module, B an A-algebra, then M is flat over A iff M_P is flat over $A_{\mathfrak{p}}$, where $P \cap A = \mathfrak{p}$. In particular we can apply to when $B = A$.
- Let $f : A \to B$, the fibre of $\operatorname{Spec} f$ over $\mathfrak{p} \in \operatorname{Spec} A$ corresponds to $\operatorname{Spec} B \otimes \kappa(\mathfrak{p})$.
  - Note that $B \otimes \kappa(\mathfrak{p}) = B \otimes A/\mathfrak{p} \otimes A_{\mathfrak{p}} = (B/\mathfrak{p}B)_{\mathfrak{p}} = (B/\mathfrak{p}B)_{f(A \backslash \mathfrak{p})}$. Therefore, $\operatorname{Spec} B \otimes \kappa(\mathfrak{p})$ contains exactly those prime ideals that contain $f(\mathfrak{p})$ and does not intersect $f(A \backslash \mathfrak{p})$.
- M is flat iff $I \otimes M \to IM$ is an isomorphism for each ideal I.
  - This implies that over a PID A, a module is flat iff it is torsion-free:

$$\begin{array}{ccc} A \otimes M & \longrightarrow & I \otimes M \\ \downarrow & \swarrow & \\ IM & & \end{array}$$

  - Submodules of flat modules are not necessarily flat. For example, $I \to A$ is always injective, but $I \otimes_A I \to I \otimes A = I$ might not be.
- Let M flat, then tensoring with M preserves kernel and image: consider tensoring M with

$$0 \longrightarrow \ker f \longrightarrow N \longrightarrow N' \longrightarrow \operatorname{coker} f \longrightarrow 0$$

  we then get $\ker f \otimes M = \ker f_M$ and $\operatorname{coker} f \otimes M = \operatorname{coker} f_M$ (which implies $\operatorname{im} f \otimes M = \operatorname{im} f_M$).
- A Dedekind domain is an ID where localizations at prime ideals are PIDs. This and the localness of flatness implies that an A-module is flat iff it is torsion-free.
  - Torsion-freeness is also a local property.
- (Faithfully Flatness): A flat A-module M is faithfully flat if one of the followings holds:

- $M \neq \mathfrak{m}M$ for every $\mathfrak{m}$.
- $M \otimes N = 0$ implies $N = 0$.
- $N_1 \otimes M \to N_2 \otimes M$ isomorphism implies $N_1 \to N_2$ isomorphism.

- Examples: A_S is flat over A but not faithfully flat, unless S contains only units. $k[x]$ is not flat over $k[x^2, x^3]$.
- A ring map is faithfully flat iff it induces surjection on Spec or mSpec.
- Another property for flat modules: If B flat A-module, M f.p. A-module, N A-module, then

$$\mathrm{Hom}(M, N) \otimes B \to \mathrm{Hom}(M \otimes B, N \otimes B)$$

is an isomorphism.

- Minimal basis and basis are in general different (a basis has to be linearly independent). However, if A is local and M is finite flat A-module, then minimal basis in M corresponds to basis in $M/\mathfrak{m}M = k^n$, and is itself a basis. Therefore over a local ring free, flat, and faithfully flat are the same if the module is finitely generated.
- A f.p. module is flat iff it is projective.

1.2.1 Pure Submodules

- $N \subset M$ is a pure submodule if $N \otimes E \to M \otimes E$ is injective for every A-module E. Equivalently, if a system of linear equations $\sum a_{ij}x_j = n_i$ has solutions in M, then it also has solutions in N.
- Geometrically, N is a pure submodule of M if not only as quasi-coherent sheaves the fibres are subsets, but also this holds up to any base change.

1.3 Chain Conditions

- For modules, Noetherian and Artinian are not related ($W = \{\frac{a}{p^n} \in \mathbb{Q}\}$ is not Noetherian but is Artinian). For rings, Artinian implies Noetherian:
  - An Artinian ring has a finite number of maximal ideals. Let $I = \mathfrak{p}_1 ... \mathfrak{p}_\mathfrak{n}$, we have a composition series: $A \supset \mathfrak{p}_1 \supset ... \supset I \supset I\mathfrak{p}_1 \supset ... \supset I^s = 0$.
  - An Artinian ring is a $\dim 0$ Noetherian ring, where we show the local case first.
- In fact, if an A-module M is Noetherian, then $A/\operatorname{ann}(M)$ is a Noetherian ring.
  - WLOG assume M faithful. Assume $M = Ax_1 + ... + Ax_n$. $A = \{(ax_1, ..., ax_n)\} \subset M^n$ is a submodule and is Noetherian.

1.4 Completion

- Given an Abelian group G, we can make G a topological group by considering filtration (G_n), where $G_0 \supset G_1 \supset ...$.
  - Note that to turn G into a topological group, it suffices to define a basis at 0 and translate it to every $g \in G$, where the basis satisfies that for every V, there exists $V_1 - V_2 \subset V$ in the basis.
  - Two filtration define the same topology if they are "included" in each other.
  - We can do the same for rings and modules. In particular, we have I-adic topology on A and M by considering (I^n) and $(I^n M)$.

  - G is Hausdorff iff $\bigcap G_n = 0$.

- The completion of G to be $\hat{G} = \lim G/G_n$, where $G/G_n \to G/G_{n-1}$ are naturally induced maps. Let $\hat{G}_n = \{(g_m), g_m = 0 \text{ for } m \leq n\}$, then $G/G_n = \hat{G}/\hat{G}_n$. (This implies that the completion of completion does not do anything.)
  - A completion is complete and Hausdorff, and uniquely factors through any map $G \to K$, K complete and Hausdorff.
  - Examples include: p-adic completion $\mathbb{Z}_p$; $\mathfrak{n}$-adic completion of $A[x_1, ..., x_n] = A[[x_1, ..., x_n]]$.

- Let $H \subset G$ be a subgroup. In general $\hat{H} = \overline{\phi(H)}$, and it is not true that $\hat{G} \to \widehat{G/H}$ is surjective. However, this is true if the index set is $\mathbb{Z}_+$.

- (Hensel's lemma): Suppose $(A, \mathfrak{m}, k)$ is $\mathfrak{m}$-adic complete. Suppose $F \in A[x]$ can be factored into gh in $k[x]$, where $(g, h) = 1$, then there exists $G, H \in A[x]$ such that $F = GH$, $\bar{G} = g$, $\bar{H} = h$.
  - We sort of construct G_n, H_n inductively such that $F - G_n H_n \in \mathfrak{m}^n$. Now, completeness guarantees that $G = \lim G_n$ and $H = \lim H_n$ exist.

- (Artin-Rees): Let A Noetherian, M f.g., $N \subset M$, then there exists n_0 such that $n \geq n_0$ implies:

$$(I^{n+1}M \cap N) = I(I^n M \cap N).$$

  - This follows from the more general fact that the restriction of a stable I-filtration is still stable (given finiteness conditions).
  - A corollary: given the finiteness conditions, M is complete iff A is complete w.r.t. the I-adic topology.
  - Let A Noetherian, the I-adic completion $\hat{A}$ is flat over A.

- A special example: Let A be the ring of real analytic functions at 0, then the (x)-completion of A is $\mathbb{R}[[x]]$.
  - Note that $A/(x^n) = \mathbb{R}[x]/(x^n)$, and $\mathbb{R}[[x]]$ is exactly the (x)-completion of $\mathbb{R}[x]$.

- Suppose that M is an A-module with I-adic topology. A submodule N is closed if $N = \bigcap(N + I^n M)$. Indeed, the right hand side depicts points that are arbitrarily close to N, i.e., in the closure of N.

1.5 Dimension Theory

- If $A = k[\alpha_1, ..., \alpha_n]$ satisfies that $\operatorname{tr.deg}_k A > 0$, then A is not a field. This implies that $k[x_1, ..., x_n]/\mathfrak{m}$ is an algebraic extension of k.
  - This implies that we have an embedding $\theta : k[x_1, ..., x_n]/\mathfrak{m} \to \bar{k}$.

- Let $(A, \mathfrak{m})$ be Noetherian local, $f \in \mathfrak{m}$, then $\dim A/fA = \dim A - 1$ if f is not in any minimal prime ideal (the closed irreducible sets are not altered by modulo f).

- Over Noetherian rings, transcendence degree behaves well w.r.t. dimension. i.e., $\operatorname{ht} \mathfrak{n} = \operatorname{ht} \mathfrak{m} + n$ if $\mathfrak{n}$ is a maximal ideal of $A[T_1, ..., T_n]$ whose restriction to A is $\mathfrak{m}$. In particular, $\dim A[T_1, ..., T_n] = \dim A + n$.

- If $A \to B$ is an integral extension, then $\dim A = \dim B$. In particular, for f.g. ID over k, this shows $\operatorname{tr.deg} QF(A) = \dim A$.

- For f.g. ID A over k, $\dim A/\mathfrak{p} = \dim A - 1$ for $\mathfrak{p} \in \operatorname{Spec} A$ of height 1.

- Let $\operatorname{gr} A = \bigoplus I^n/I^{n+1}$. If A is f.g., then $\operatorname{gr} A = (A/I)[\xi_1, ..., \xi_n]$ which is f.g..
- Recall that if M_n is a f.g. R_0-module, with R_0 Artinian, then M_n has finite length. We define the Hilbert series to be

$$P(M,t) = \sum l(M_n)t^n.$$

If R Noetherian with r generators over R_0, we can write $P(M,t) = \frac{f(t)}{\prod_{i=1}^r (1-t^{d_i})}$ where a generator x_i is of degree d_i.

  - A corollary, if $R = R[x_i]$ with $\deg x_i = 1$, we can express $l(M_n)$ in some polynomial $\varphi_M(n)$ for all large enough n. φ_M is the Hilbert polynomial, and $d(M) = \deg \varphi_M + 1$.
- (Samuel Function). Essentially the same as Hilbert function, except that we are working over a Noetherian semilocal ring and we pass to the graded ring and module $\operatorname{gr}_I(A)$ and $\operatorname{gr}_I(M)$ so that the generators are of deg 1, so that we can talk about the Hilbert polynomial and $d(M)$. In particular $\chi_M^I(n) = l(M/I^{n+1}M)$ and $\deg \chi_M^I = d(M)$.
- Let $\delta(M)$ be the min value $x_1, ..., x_n \in \mathfrak{m}$ = Jacobson radical of A such that $l(M/\sum x_i M) < \infty$. Still, let A be a semilocal Noetherian ring and M finite A-module. We have $\dim M = d(M) = \delta(M)$.
- Let A Noetherian. $\dim M = 0$ iff M is of finite length.
  - Recall $\dim(M) = \dim A/\operatorname{ann}(M) =: \dim B$. If B is of dim 0 then B is Artinian, so M is f.g. iff $l(M) < \infty$.
  - Conversely, suppose M is of finite length, then $\operatorname{Supp} M$ is a finite set of maximal ideals since $A/\operatorname{ann}(M)$ is Artinian, so $\dim M = 0$ since $V(\operatorname{ann} M) = \operatorname{Supp} M$.
- (Principle Ideal Theorem). Let A Noetherian and $I = (x_1, ..., x_d)$, then if $\mathfrak{p}$ is a minimal prime ideal over I, then $\operatorname{ht} \mathfrak{p} \le d$. In particular, $A_\mathfrak{p}$ is finite dimensional.
- (Regularity). Let $(A, \mathfrak{m})$ Noetherian local. Recall that if $\mathfrak{m}$ is generated by less than r elements, then $\operatorname{ht} \mathfrak{m} = \dim A < r$. Now, suppose $\dim A = r$, then A is regular if $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$, in particular, iff $\mathfrak{m}$ is generated by r elements.
  - A dim 1 regular local ring is just a DVR. In general, a regular local ring is an ID.
  - Let A dim d regular local, then $\operatorname{gr}_\mathfrak{m} A \cong k[X_1, ..., X_d]$.
  - If the graded ring $\operatorname{gr}_\mathfrak{m} A$ is non-reduced, $\operatorname{Spec} A$ is not smooth at $\mathfrak{m}$.
- Recall that $\dim B \otimes \kappa(\mathfrak{p}) = \sup_{\varphi^{-1}(P)=\mathfrak{p}} \dim B_P/PB_P$. Also, $\dim B_P/\mathfrak{p}B_P$ is the supremium of the length of chains of prime ideals $P_0 \subset ... \subset P_n = P$ such that all of them lie over $\mathfrak{p}$.
- Suppose $\varphi : A \to B$ is flat between Noetherian rings, or more generally the going-down property holds, then $\operatorname{ht} P = \operatorname{ht} \mathfrak{p} + \dim B_P/\mathfrak{p}B_P$. In particular, if A, B Noetherian local, then $\dim B = \dim A + \dim B/\mathfrak{m}B$.
- Let $\varphi : A \to B$ between Noetherian rings. If $\mathfrak{p} \supset \mathfrak{q}$ in $\operatorname{Spec} A$, and going-up (going-down) holds, then $\dim B \otimes \kappa(\mathfrak{p}) \ge \dim B \otimes \kappa(\mathfrak{q})$ ($\dim B \otimes \kappa(\mathfrak{p}) \le \dim B \otimes \kappa(\mathfrak{q})$).

1.6 Associated Prime Ideals

- Minimal elements in $\operatorname{Ass} M$ correspond to irreducible components of the support of M as a quasi-coherent sheaf; i.e., $\mathfrak{p} \in \operatorname{Ass}(M)$ corresponds to $V(\mathfrak{p})$.
  - If we consider $\operatorname{Ass}(M_S)$, this is exactly $\operatorname{Ass}(M) \cap \operatorname{Spec}(A_S)$: i.e., the support of M intersecting with the subset $\operatorname{Spec} A_S$.

- In particular, $\mathfrak{p} \in \mathrm{Ass}(M)$ iff $\mathfrak{p}A_\mathfrak{p} \in \mathrm{Ass}(M_\mathfrak{p})$.
- As a result, minimal elements of $\mathrm{Ass}(M)$ coincide with minimal elements of $\mathrm{Supp}(M)$.

- We define primary submodules of M in the same fashion as for primary ideals: if a is an annihilator of M/N, then $a \in \sqrt{\mathrm{ann}(M/N)}$.
  - Given finiteness conditions, $N \subset M$ is primary iff $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$.
  - This is saying that a primary submodule is where $\mathrm{Supp}(M/N)$ is an irreducible set (though may not be an irreducible component).
  - Such submodule N is called $\mathfrak{p}$-primary.
  - This also explains why irreducible submodules are primary.
- Explicitly, let $\mathfrak{p}_i$ be the minimal primes over I. There are $\mathfrak{p}_i$-primary ideals q_i such that $I = \bigcap q_i$.

1.7 Integral Extensions

- Suppose that $E \subset R \subset F$ and F/E is algebraic. Then R is itself a subfield.
  - For finite extension case we can consider $\times_a : R \to R$ for any $a \in R$. In general en explicit construction works.
- Let $A \to B$ be an integral inclusion. Then if B is a field, so is A. This implies that if $A \to B$ is integral, then $\mathrm{Spec}\,\varphi$ sends closed points to closed points.
  - Being an integral extension is a local property: if $A \to B$ is integral, then so is $A_\mathfrak{p} \to B \otimes A_\mathfrak{p}$.
  - Also, if φ is injective, then $B \otimes A_\mathfrak{p} = \varphi(A \backslash \mathfrak{p})^{-1}B \neq 0$. In particular, if φ is injective and integral then $\mathrm{Spec}\,\varphi$ is surjective.
- Finiteness (as a module) over a ring corresponds to integral extension.
- (Going Up/Going Down). If B/A is integral, then going-up holds. If furthermore B is an ID and A is integrally closed, then going-down holds.
  - Key for proving going-down: If A integrally closed in its field of fractions K. L a normal field extension of F, then if B is the integral closure of A in L, then all prime ideals in B lying over $\mathfrak{p}$ are conjugates over K.
  - This creates, say given $P' \in \mathrm{Spec}\,B$ over $\mathfrak{p}'$, a $P \in \mathrm{Spec}\,B$ over $\mathfrak{p} \subset \mathfrak{p}'$, where we consider a larger extension ring that contains B and is the integral closure of A in some normal extension of $K = QF(A)$, and apply going-up theorem to the integral extension C/B.
- Flatness gives going-down theorem.
- If B/A is integral, then so is $B[x_1, ..., x_n]/A[x_1, ..., x_n]$.

1.8 Valuation Rings

- If $A \subset K$ is a subring, then $B = \cap R$, where $R \supset A$ valuation rings, is the integral closure of A.
- Let $A \subset K$ valuation ring. We have a canonical valuation: $v : K \to G \cup \{\infty\}$ with $G = \{xR, x \in K^\times\}$, where we define $v(0) = \infty$ and $xR \leq yR$ iff $xR \supset yR$. We then have $R_v = \{x \in K, v(x) \geq 0\} = R$. This is not unique but unique up to some order preserving isomorphism on the codomain.

- Characterization for DVR: dim 1 normal Noetherian local ring. For Dedekind domain: a field or a dim 1 Noetherian normal domain, or if every ideal is a finite product of prime ideals, or if all localizations at prime ideals are DVRs.
- One way for constructing Dedekind domain: let A dim 1 Noetherian ID, $K = QF(A)$, L/K algebraic, $B = \bar{A}$ in L, then B is Dedekind.
- Suppose A normal Noetherian ID, then $A = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} A_\mathfrak{p}$.
- A valuation ring or a Dedekind domain is a Prüfer domain: a module over the ring is flat iff torsion-free.
- A Krull ring A satisfies $A = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} A_\mathfrak{p}$, and that each $A_\mathfrak{p}$ DVR, and that every $x \in A$ lives in at most a finite number of $\mathfrak{p}$ with height 1.
  - Essentially, this is saying that $\operatorname{Spec} A$ is reconstructed from codim 1 data, and that principal divisor makes sense.
  - A Dedekind domain is exactly a dim 1 Krull ring.
  - As noted earlier, a normal Noetherian ID is a Krull ring.
- Suppose $A \to B$ turns B to a finite A-module, then fibers of the spectrum map is finite. The fiber is given as $\operatorname{Spec} B \times_{\operatorname{Spec} A} \{x\}$, which corresponds to $\operatorname{Spec} B \otimes_A k$ and this is a finite k-vector space.

1.9 Regular Sequences

- (Regular and Quasi-Regular Sequence). Let M A-module, $(a_1, ..., a_n)$ is a M-sequence if $a_i x \neq 0$ for any $x \in M/\sum_{j<i} a_j M$ and that $M/\sum a_i M \neq 0$. Let $I = \sum a_i A$. The sequence is a M-quasi-regular sequence if for any $F \in M[X_1, ..., X_n]$ of homogeneous degree ν, $F(a) \in I^{\nu+1}M$ implies all coefficients of F are in IM.
  - Note that $F(a)$ denotes an element in $I^\nu M$, so this is saying if $\sum x_i m_i \in I^{\nu+1}M$ where $x_i \in I^\nu M$, then $m_i \in IM$.
  - M-sequences are M-quasi-regular sequences.
  - A sequence is M-quasi-regular iff the map $\varphi : M/IM[x_1, ..., x_n] \to \bigoplus I^\nu M/I^{\nu+1}M$ is an isomorphism, where f homogeneous of degree ν is sent to the class of $F(a)$ in $I^\nu M/I^{\nu+1}M$.
- Geometrical meaning of regular sequences: $f \in A$ is a non-zero divisor iff $f \notin \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Ass}(A)$. Correspondingly, $\dim X_j \cap V(f) < \dim X_j$ for each associated component X_j. Generalizing this, a sequence is regular if for each i, $\dim X_j \cap V(f_1) \cap ... \cap V(f_i) < \dim X_j \cap V(f_1) \cap ... \cap V(f_{i-1})$ for each associated component.
  - We later see that for A Noetherian local, $\dim A/\mathfrak{p} \geq \operatorname{depth} M$ if $\mathfrak{p} \in \operatorname{Ass}(M)$. Indeed, consider the associated component $V(\mathfrak{p})$. A regular sequence of length r in $\mathfrak{m}$ means that we can gradually reduce the dimension of $V(\mathfrak{p})$, each step by at least 1. This implies $\dim V(\mathfrak{p}) \geq \operatorname{depth}(M)$.
  - Essentially, a system of parameter reduces the dimension by 1 each step, and a regular sequence reduces the dimension of each associated component by at least 1 at each step.
  - More generally, $\dim M = \sup \dim A/\mathfrak{p} \geq \inf \dim A/\mathfrak{p} \geq \operatorname{depth} M$.
- (Koszul Complex). Let $x_1, ..., x_n \in A$, we define $K_p = \bigoplus A e_{i_1,...,i_p}$ for $p \leq n$, and the boundary maps $d(e_{i_1,...,i_p}) = \sum (-1)^{r-1} x_{i_r} e_{i_1,...,\hat{i}_r,...,i_p}$ and the homology groups correspondingly. In particular, $H_0(\underline{x}, M) = M/\sum x_i M$ and $H_n(\underline{x}, M) \cong \{\xi, x_i\xi = 0\}$. In general $(x_1, ..., x_n)$ annihilates all $H_p(\underline{x}, M)$. If $x_1, ..., x_n$ is regular then $H_p(\underline{x}, M) = 0$ for all p.

- Let A Noetherian and M finite A-module. $IM \neq M$. $\mathrm{Ext}_A^i(N, M) = 0$ for all finite A-module N with $\mathrm{Supp}(N) \subset V(I)$ and $i < n$ iff there exists an M-sequence of length n in I.
- In fact, we have $n =: \mathrm{depth}(I, M) = \inf\{i, \mathrm{Ext}^i(A/I, M) \neq 0\}$. For Noetherian local ring A, we write $\mathrm{depth}\, M = \mathrm{depth}(\mathfrak{m}, M)$.
  - In particular, suppose A Noetherian local, M and N finite A-modules, then $\mathrm{Ext}_A^i(N, M) = 0$ for $i < k - r$, where $k = \mathrm{depth}\, M$ and $r = \dim N$.
- Let A Noetherian, M finite A-module with $M \neq IM$, where $I = (y_1, ..., y_n)$, then $\mathrm{depth}(I, M)$ is the number of successive zero terms from the left in $H_n(\underline{y}, M), H_{n-1}(\underline{y}, M), ..., H_0(\underline{y}, M)$.

1.10 Cohen-Macaulay Rings

- Essentially this is some sort of equi-dimensional condition: in general $\mathrm{depth}\, M \leq \dim M$, and M is a CM module if $\mathrm{depth}\, M = \dim M$. This implies that M has no embedded associated primes. A CM local ring A satisfies $\mathrm{ht}\, I + \dim A/I = \dim A$.
- Equivalently, A is a CM ring if all ideals I satisfy unmixed condition, i.e., $\mathrm{Ass}\, A/I = \min A/I$, or if there is no embedded prime divisors of I.
- A ring is CM iff the regular sequences are the same as (part of) systems of parameter.
- Note that if A is a Noetherian local ring, then a maximal regular sequence in $\mathfrak{p}$ extends to a maximal regular sequence in $\mathfrak{m}$. In particular, $\mathrm{depth}\, A/(x_1, ..., x_r) = \mathrm{depth}\, A - r$ if $(x_1, ..., x_r)$ is a regular sequence.

2 Algebraic Geometry

2.1 Sheaves and Ringed Spaces

- Stalk-wise injectivity/surjectivity for presheaf maps does not imply section-wise. On the other hand, stalk-wise injectivity implies section-wise injectivity for sheaf maps.
  - In fact, we might just define surjective (pre)sheaf maps to be those surjective on stalks.
- $\mathcal{F}$ is a sheaf iff the sequence $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j)$ is exact.
- Two ways of sheafification:
  - $\mathcal{F}^+(U) = \{f : U \to \coprod \mathcal{F}_x \text{ locally compatible}\}$.
  - $\mathcal{F}^+(V) = \mathrm{colim}_{\mathcal{V}}\, \mathcal{F}_{\mathcal{V}}(V)$, where $F_{\mathcal{V}}$ is the kernel of the above sequence.
- Gluing sheaves: let $\mathcal{F}_i$ be a sheaf on U_i, where (U_i) open cover. Let (φ_{ij}) isomorphism between $\mathcal{F}_i|_{U_i \cap U_j}$ and $\mathcal{F}_j|_{U_i \cap U_j}$ satisfying the cocycle conditions. There exists a gluing $\mathcal{F}$ such that $\phi_i : \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ isomorphisms with $\varphi_{ij} \circ \phi_i = \phi_j$.
- (Support of a sheaf): $\mathrm{Supp}\, \mathcal{F} = \{x, \mathcal{F}_x \neq 0\}$. This is in general not closed.
  - If $\mathrm{Supp}\, \mathcal{F}$ is a finite number of closed points, then $\mathcal{F}$ is a skyscraper sheaf, with $\mathcal{F}(U) = \bigoplus_{x \in U} \mathcal{F}_x$. (Consider $\mathcal{F}(U)$ as the locally compatible tuples in $\prod_{x \in U} \mathcal{F}_x$).
- We can glue morphisms between ringed spaces in the sense that if $f_i : U_i \to Y$ morphisms that agree on intersections, we have unique $f : X \to Y$.
- (Sheaf of ideal): $\mathcal{J}$ is a sheaf of ideal if $\mathcal{J}(U) \subset \mathcal{O}_X(U)$ is an ideal for every U. Let $V(\mathcal{J}) = \{x \in X, \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$ with structure sheaf $j^{-1}(\mathcal{O}_X/\mathcal{J})$.

- A morphism $f : X \to Y$ between ringed spaces is closed/open immersion if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective/isomorphism.
  - $j : V(\mathcal{J}) \to X$ is a closed immersion.
  - We can factor a closed immersion into an isomorphism with $V(\mathcal{J})$ followed by the canonical immersion.
- (Quotient of ringed spaces): let G acts on X. We define $X/G =: Y$ by $\mathcal{O}_Y(U) = (\mathcal{O}_X(p^{-1}(U))^G$.
  - For schemes quotient might not exists (and can be different from the quotient as ringed spaces).

2.2 Schemes

- Consider $f : \operatorname{Spec} \mathbb{Z}[t] \to \operatorname{Spec} \mathbb{Z}$. $\operatorname{Spec} \mathbb{Z}[t] = f^{-1}(\{0\}) \cup \bigcup f^{-1}(p\mathbb{Z})$.
  - If $\mathfrak{p} \in f^{-1}(\{0\})$, then $\mathfrak{p} \cap \mathbb{Z} = \{0\}$. In particular, we can consider the localization $S^{-1}\mathfrak{p}$ in $S^{-1}\mathbb{Z}[t] = \mathbb{Q}[t]$, where $S = \mathbb{Z} \backslash \{0\}$. Therefore, $f^{-1}(\{0\})$ correspond to $\operatorname{Spec} \mathbb{Q}[t]$.
  - If $\mathfrak{p} \in f^{-1}(p\mathbb{Z})$, then $(p) = \mathbb{Z} \cap \mathfrak{p}$. We can then correspond $\mathfrak{p}$ with a prime ideal in $\mathbb{Z}/(p)[t] = \mathbb{F}_p[t]$. Therefore, $f^{-1}(\{p\})$ correspond to $\operatorname{Spec} \mathbb{F}_p[t]$.
  - We conclude that $\operatorname{Spec} \mathbb{Z}[t] = \operatorname{Spec} \mathbb{Q}[t] \cup \bigcup \operatorname{Spec} \mathbb{F}_p[t]$, a family of affine lines.
- A scheme is a ringed space that is also locally affine schemes.
- Let $X_f = \{x \in X, f_x \in \mathcal{O}_{X,x}^\times\}$. If X qcqs, then $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ is an isomorphism.
- A closed subscheme consists of a closed immersion $j : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$. Affine locally, j is the same as $\operatorname{Spec} A/J \to \operatorname{Spec} A$.
  - For this one, we show qcqs of Z if X affine, and move to affine opens to conclude that $\ker j^\# = \ker i^\#$ where $i : \operatorname{Spec} A/J \to \operatorname{Spec} A$ (so $Z \cong \operatorname{Spec} A/J$ is compatible with the immersions).
- Given Y affine $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$.
  - If X is also affine, $f : X \to Y$ induces $\varphi : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$, which further induces $f_\varphi : X \to Y$, and we show that $f = f_\varphi$ by showing that $f^\# = f_\varphi^\#$ agree on stalks.
  - For general case, glue the morphisms $U_i \to Y$ on affine locals.
- A scheme over $\operatorname{Spec} A$ is equivalent to a scheme with structure sheaf of A-algebras.
- Let X a scheme over k. A k-rational point of X corresponds to sections $\operatorname{Spec} k \to X$. This is equivalent to picking $x \in X$ such that $\kappa(x) = k$.
  - For example, if $X = \operatorname{Spec} k[T_1, ..., T_n]/I$, then k-rational points correspond to maximal ideals with residue field k. This also corresponds to points in k^n that annihilate functions in I (if k algebraically closed).
  - Similarly, points in $\mathbb{P}^n$ corresponds to k-rational points in $\mathbb{P}_k^n$ via the identification sending $[\alpha_0 : ... : \alpha_n] \to I = (\alpha_i T_j - \alpha_j T_i)$. Similarly, this corresponds to maximal ideals with residue field k.
  - Suppose $I = (P_1, ..., P_m)$, then k-rational points of $\operatorname{Proj} k[T_0, ..., T_n]/I$ correspond to $Z_+(P_1, ..., P_m) \subset \mathbb{P}^n$.
- Let B graded. $\operatorname{Proj} B$ has basis $D_+(f)$, where f homogeneous, such that $D_+(f) \cong \operatorname{Spec} B_{(f)} = \{\frac{a}{f^n}, \deg a = n \cdot \deg f\} \subset B_f$.
- For example, $\mathbb{P}_A^n = \operatorname{Proj} A[x_1, ..., x_n]$ with $B_{(T_i)} = A[\frac{T_1}{T_i}, ..., \frac{T_n}{T_i}]$.

- Embedding $\operatorname{Spec} k[\alpha_1, ..., \alpha_n]$ as a dense open subscheme of a projective variety: consider $\operatorname{Proj} k[\alpha_1, ..., \alpha_n, X]$, with $\deg X = 1$ and $\deg \alpha_i = 0$.

- Recall that quasi-coherent sheaves of ideals correspond to closed subschemes of a scheme X. Now, let $f : X \to Y$ quasi-compact, let $\mathcal{J} = \ker f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, and let $Z = V(\mathcal{J})$, then Z is a scheme:

  - It suffices to show that $\mathcal{J}$ is quasi-coherent. Now,

$$\begin{aligned}\mathcal{J}(D(g)) &= \ker \mathcal{O}_Y(D(g)) \to \mathcal{O}_X(f^{-1}(D(g)) \\ &= \ker A_g \to \mathcal{O}_X(\operatorname{Spec} B_i \cap f^{-1}(D(g))) \\ &= \ker A_g \to \prod B_{i,\phi_i(g)} \to \prod B_{i,\phi_i(g)f_{ij}}\end{aligned}$$

    where φ_i induces $U_i \to U$. On the other hand, due to the *quasi-compactness* of f, we can commute finite product with localizations and conclude that $\mathcal{J}(U)_g$ equals the same kernel.

  - In fact, Z is the scheme-theoretic closure of $f(X)$ in Y in the following sense: there exists $g : X \to Z$ that factors f, and for any other factorization, there exists a natural $Z \to Z'$ closed immersion that factors the factorization through this one.

- The categorical quotient in the category of schemes might not be geometrically meaningful: for example, let $\mathbb{G}_m$ acts on $\operatorname{Spec} k[t]$ by $\lambda \cdot t = \lambda t$. Say k is algebraically closed. Then $\operatorname{Spec} k[t]/\mathbb{G}_m = \operatorname{Spec} k$ is a single point, while there are two orbits. It is not the orbit space in general.

- In general we might be collapsing several orbits into one point. See Mumford's work on geometric invariant theory for more information.

2.3 Reduced & Integral Schemes

- X is reduced if $\mathcal{O}_{X,x}$ is reduced for every $x \in X$. For any X, we define an associated reduced closed subscheme $X_{\text{red}} = (X, \mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}(U) = \{s \in \mathcal{O}_X(U), s_x = 0, \forall x \in X\}$.

- Let X be an algebraic variety over a field k. For $f \in \mathcal{O}_X(X)$, we define $\tilde{f} : X^0 \to \bar{k}$ sending x to $f_x \in \kappa(x) \subset \bar{k}$. This defines $\mathcal{O}_X(X) \to F(X, k) = \operatorname{Hom}(X^0, \bar{k})$ a group homomorphism.

  - Note that $\kappa(x)$ might not be a subfield of $\bar{k}$ if x is not closed. For example, if x is the generic point of $\operatorname{Spec} k[x]$, then $\kappa(x) = k(T)$ is transcendental over k.
  - In general, this gives $\mathcal{O}_X \to \mathcal{F}_X$, and the kernel is exactly the sheaf of ideals $\mathcal{N}$. In other words, this map is injective iff X is reduced.

- X is integral if it is reduced and irreducible. This implies that $\mathcal{O}_{X,x}$ is an integral domain for every x.

  - If X is integral, then $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$ are injective. In particular, $\mathcal{O}_X(U) = \bigcap \mathcal{O}_{X,x}$.
  - $K(X) = \mathcal{O}_{X,\xi}$ is called the ring of rational functions on X. f is regular at x if $f \in \mathcal{O}_{X,x}$.

2.4 Morphisms

- Fiber over y for $f : X \to Y$: $X_y = X \times_Y \operatorname{Spec} \kappa(y)$. This is topologically homeomorphic to $f^{-1}(y)$. We can consider $X_y \to \operatorname{Spec} \kappa(y)$ as a family of schemes over $\kappa(y)$ parametrized by y.

- Let X algebraic variety over k. For K/k algebraic, we denote X_K to be the base change via $\operatorname{Spec} K \to \operatorname{Spec} k$.

  - $\dim X_K = \dim X$.
  - If K/k separable and X reduced, so is X_K.
  - If K/k purely inseparable, $X_K \to X$ is a homeomorphism.

- (Separated Morphisms). X is separated over Y if $X \to X \times_Y X$ is a closed immersion, or if $\Delta(X)$ is closed. X is separated iff for every pair of open affines U, V, $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.
  - Essentially, because a closed immersion is affine and can be verified Zariski-locally.
  - The Valuative Criterion: Let $f : X \to Y$ quasi-separated. f is separated/universally closed/proper if for every $\operatorname{Spec} V \to Y$ and $\operatorname{Spec} K \to X$ Y-morphism where V valuation ring and $K = QF(V)$, the corresponding map $X_Y(V) \to X_Y(K)$ is injective/surjective/bijective.

- (Proper Morphisms). A map $f : X \to Y$ is universally closed if f is closed and so is any base change of f. $f : X \to Y$ is proper if it is of finite type, separated, and universally closed. This can also be verified locally.

- For affine schemes, proper implies finiteness.
  - By Artin-Tate & Noether Normalization: B is a f.g. A-module if it is a f.g. A-algebra and is integral over A.
  - Or more easily one can directly prove the above proposition by considering the maximal power x_i^d such that x_i^d not generated by $(x_i, ..., x_i^{d-1})$ over A for each generator x_i.
  - More generally, if $X/\operatorname{Spec} A$ is proper, then $\mathcal{O}_X(X)$ is integral over A.

- Let X proper over $\operatorname{Spec} \mathcal{O}_K$, then $X_K(K) = X(\mathcal{O}_K)$.

- Any projective morphism is proper. Also, due to the Segre embedding, the family of projective morphisms satisfies the "good" properties. The same properties hold for quasi-projective morphisms (decomposed to an open immersion and a projective morphism).

- (Chow's Lemma). Let Y Noetherian, any $X \to Y$ proper is a projective morphism after modifying outside of a dense open subset.

- In fact, separated morphisms can be detected on the universal reduced closed subscheme: $X \to Y$ is separated iff $X_{\text{red}} \to Y_{\text{red}}$ is. The analogous holds for proper morphisms, assume the map is of finite type to begin with.

- Finite and integral morphisms are affine local. Finite morphisms are of finite type, quasi-finite, and proper. Also, finite morphisms between affine schemes is projective.

2.5 Normal Schemes

- (Normal Scheme). X is normal if X is irreducible and $\mathcal{O}_{X,x}$ is normal for every x.
  - Let X quasi-compact, then it suffices to detect normality at closed points. (Note that for every x there exists $y \in \{x\}$ closed, and $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$.
  - Similarly, reducedness, dimension, or a sheaf $\mathcal{F} = 0$ are detectable at stalks of closed points (assume quasi-compactness).
  - X is normal implies that X is reduced. So X is integral.

- A Dedekind domain is a normal Noetherian ID of $\dim \leq 1$, and a Dedekind scheme is a normal locally Noetherian scheme of $\dim \leq 1$. (When $\dim = 0$, we only have fields.)

- Normality implies unique extension of regular functions when the codimension is large: $\mathcal{O}_X(X) \to \mathcal{O}_X(X \backslash F)$ is an isomorphism if F is of codimension ≥ 2.

  - This essentially follows from $A = \bigcap A_{\mathfrak{p}}$ over all ht 1 prime ideals.

- Let X integral. The normalization of X is a morphism $X' \to X$ with X' normal, such that any $Y \to X$ dominant with Y normal factors through X'. For example, $\operatorname{Spec} \bar{A} \to \operatorname{Spec} A$ is a normalization morphism.

  - Let X integral. A morphism $f : Y \to X$ is a normalization iff Y normal, f birational and integral.

2.6 Regular Schemes

- The tangent space $T_{X,x}$ is defined as the dual of the $\kappa(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. We have $\dim T_{X,x} \geq \dim \mathcal{O}_{X,x}$.

  - For example, let $X = \operatorname{Spec} k[x_1, ..., x_n]$, then we define $D_x : k[x_1, ..., x_n] \to (k^n)^\vee$, where $D_x P$ sends $(t_1, ..., t_n)$ to $\sum \frac{\partial P}{\partial x_i}(x) t_i$. By considering the Taylor expansion, we see that for $\mathfrak{m} = (x_1 - \lambda_1, ..., x_n - \lambda_n)$, we have $\mathfrak{m}/\mathfrak{m}^2 = (k^n)^\vee$ and that $T_{X,x} \cong k^n$.
  - Similarly, let $\mathfrak{m}$ be a maximal ideal in $k[T_1, ..., T_n]$. $\varphi_x : \mathfrak{m}/\mathfrak{m}^2 \to (\kappa(x)^n)^\vee$ is an isomorphism iff $\kappa(x)/k$ is separable.
  - If $X = V(I) \subset \mathbb{A}^n$, then $T_{X,x} = \{(t_1, ..., t_n) \in k^n, \sum \frac{\partial P}{\partial x_i}(x) t_i = 0 \text{ for every } P \in I\}$; i.e., the tangent space is the subspace of directions where the directional derivative along them for any $P \in I$ is zero. In particular, $T_{X,x} = (D_x I)^\perp$.
  - Recall that $(A, \mathfrak{m})$ is regular iff $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim A$, or iff $\mathfrak{m}$ is generated by $\dim A$ elements. We call a point x as a regular point if $\mathcal{O}_{X,x}$ is regular, and a singular point otherwise.
  - In particular, a Noetherian connected scheme of $\dim 1$ is normal iff it is regular.
  - A system of parameters is similar to a coordinate system: if $(A, \mathfrak{m})$ Noetherian regular local of $\dim d$, then a system of parameters is a system of d generators for $\mathfrak{m}$.
  - For example, $k[x, y]/(x^2 - y^3)$ is not regular at $(0, 0)$. In fact, if $(A, \mathfrak{m})$ is a regular Noetherian local ring, then A/fA is regular iff $f \notin \mathfrak{m}^2$. Here, $x^2 - y^3 \in (x, y)^2$ so it fails to be regular.
  - More generally, A/I is regular iff I is generated by r elements of a system of parameters of A, where $r + \dim A/I = \dim A$. The particular case above says that f is in a system of parameter iff $f \notin \mathfrak{m}^2$.

- Regular Noetherian local rings are normal and regular at localizations.

- Let $X = V(I)$ with $I = (f_1, ..., f_r)$. Note that $\dim T_{X,x} = n - \dim D_x I$, where $\dim D_x I$ is the rank of the Jacobian by the definition. So, X is regular at x iff the rank of $J_x = (\frac{\partial F_i}{\partial x_j}(x))$ is $= n - \dim \mathcal{O}_{X,x}$.

- Let X normal. $\operatorname{Reg} X$ is open and $\operatorname{codim}(X_{\text{Sing}}, X) \geq 2$.

- Let X algebraic variety over k and regular at $x \in X(k)$, then $\hat{\mathcal{O}}_{X,x} \cong k[[x_1, ..., x_d]]$ where $d = \dim \mathcal{O}_{X,x}$, where the completion is $\mathfrak{m}_x$-adic.

- Every rational point is closed, every closed point can be viewed as a geometric point. Smooth points are always regular.

2.7 Flatness and Smoothness

- Flatness is the property that assures "dimensions are right". In general, for a map between locally Noetherian schemes, $\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$, and equality holds if f is flat.

  - Essentially, if $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ with A reduced Noetherian local, $t \in A$ not nilpotent nor unit, we have $\dim A/tA = \dim A - 1$ and $\dim B/tB \geq \dim B - 1$, where equality holds if flatness is given. Now the result follows by induction.
  - For flat morphisms between irreducible algebraic varieties, the dimension of the fibers is constant, $\dim X_y = \dim X - \dim Y$.
  - Blowup fails to be flat at the points that are blowed up.

- A morphism of finite type between locally Noetherian schemes is unramified at x if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ satisfies $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$, and if $\kappa(y) \to \kappa(x)$ is separable. f is étale at x if it is unramified and flat at x.
  - For example, $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale if L/K is separable.
  - f being unramified is equivalent to X_y being finite, reduced, and $\kappa(x)/\kappa(y)$ separable.
  - $\operatorname{Spec} k[T]/(P) \to \operatorname{Spec} k$ is étale at x if x (representing $Q \mid P$) is simple in P and separable.

- Let $f : X \to Y$ étale, then $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$ with X_y finite, and $T_{X,x} \cong T_{Y,y} \otimes_{\kappa(y)} \kappa(x)$.

- X over $\operatorname{Spec} k$ is smooth at x if the points lying over x in $X_{\bar{k}}$ are all regular. If X is an algebraic variety with x closed, then X is regular at x if it is smooth at x, and the converse holds if $\kappa(x)/k$ is separable.
  - In general this would still be true, as J_x and $J_{x'}$, and $\dim \mathcal{O}_{X,x}$ and $\dim \mathcal{O}_{X_{\bar{k}},x'}$ are independent of the choice of x'.

- Let X and Y locally Noetherian and $f : X \to Y$ of finite type. f si smooth at $x \in X$ if it is flat at x and $X_y \to \operatorname{Spec} \kappa(y)$ is smooth. Note that it suffices to check smoothness at closed points of X_y over $y \in Y$ closed. f is smooth of relative dimension n if the non-empty fibers are equi-dimensional of $\dim n$, and an étale morphism is smooth of relative $\dim 0$.

- Let $\mathfrak{n}_x$ be the maximal ideal of $\mathcal{O}_{X_y,x}$, we have an exact sequence of $\kappa(x)$ vector spaces:

$$0 \longrightarrow T_{X_y,x} \longrightarrow T_{X,x} \longrightarrow T_{Y,y} \otimes \kappa(x) \longrightarrow 0$$

  In particular, étale morphisms induce isomorphisms on the tangent spaces.

2.8 Zariski's Main Theorem

- (Zariski's Main Theorem): Let Y normal locally Noetherian integral, $f : X \to Y$ separated quasi-finite birational morphism of finite type, then f is an open immersion.
  - As for the motivation, the initial version is that: if Y is normal, $f : X \to Y$ birational, and $y \in Y$ with isolated point $x \in X_y$, then the birational morphism is an isomorphism in a neighborhood of x.
  - We can "compactify" quasi-finite morphisms to finite morphisms. Now, the theorem is saying that after compactification, we assume f is finite; now since Y is normal, f must be an open immersion (locally, it is finite and therefore an open immersion, and we use the fact that a locally open immersion is an open immersion for f separated and X, Y irreducible).

- The theorem implies that the set of points where f is étale at is open if f is locally of finite type.

- Also, if Y Dedekind with generic point η, then if $f : X \to Y$ dominant of finite type, with X irreducible, then X_s is empty or of dimension equals $\dim X_\eta$.

2.9 Coherent Sheaves

- A quasi-coherent sheaf $\mathcal{F}$ is an $\mathcal{O}_X$-module such that for each $x \in X$, there exists a neighborhood $U \ni x$ and an exact sequence of $\mathcal{O}_X$-modules:

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0.$$

  In some sense quasi-coherent sheaves correspond to modules with a presentation. Recall that there exists a surjective $\mathcal{O}_X^{(I)} \to \mathcal{F}$ iff $\mathcal{F}$ is generated by I (i.e., $I = \{s\} \subset \mathcal{F}(X)$ such that $\{s_x\}$ generates $\mathcal{F}_x$ for each x.

- For a quasi-coherent sheaf, if X is Noetherian or separated & quasi-compact, then for $f \in \mathcal{O}_X(X)$, $\mathcal{F}(X)_f \cong \mathcal{F}(X_f)$. In fact, $\mathcal{F}$ is quasi-coherent iff for every open affine U, $\mathcal{F}(U)^\sim \cong \mathcal{F}|_U$.

- A coherent sheaf is an $\mathcal{O}_X$-module that is finitely generated (locally, we have surjective $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$), and if the kernel for every homomorphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ is finitely generated. The notion of coherent sheaves correspond to finitely presented modules; if X locally Noetherian, then $\mathcal{F}$ is coherent iff it is finitely generated, iff for every U open affine, $\mathcal{F}(U)$ is f.g. over $\mathcal{O}_X(U)$.

- Given an $\mathcal{O}_Y$-module $\mathcal{G}$, we can pull it back via f by $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ to get a $\mathcal{O}_X$-module.

- We define quasi-coherent sheaves on projective schemes via the following: let B be a graded ring, we define $B(n)$ to be the graded ring with $B(n)_d = B_{n+d}$ and $\mathcal{O}_X(n) = (B(n))^\sim$, with $\mathcal{O}_X(n)(D_+(f)) = f^n B_{(f)}$, where $M_{(f)} = \{\frac{a}{f^n}, \deg a = n \deg f\}$.

  - We can also define the twist of a quasi-coherent sheaf $\mathcal{F}$ by $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$.
  - We can easily see that $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$. Also, $\mathcal{O}_X(n)(X) = B_n$.
  - Let $\mathcal{F}$ f.g. quasi-coherent sheaf over a projective scheme X, there exists $n_0 \geq 0$ such that $\mathcal{F}(n)$ is f.g. by global sections if $n \geq n_0$.

- If $X = \operatorname{Proj} B$, then any quasi-coherent sheaf $\mathcal{F}$ satisfies $\mathcal{F} \cong (\bigoplus \mathcal{F}(n)(X))^\sim$.

- Let $i : X \to \mathbb{P}_A^d$ be an immersion, the sheaf $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}_A^d}(1)$ is called a very ample sheaf.

- Let $Y = \operatorname{Proj} A[T_0, ..., T_d]$, X a scheme over A. Morphisms $X \to Y$ over $\operatorname{Spec} A$ correspond to invertible sheaves $\mathcal{L}$ over X generated by $d+1$ global sections, via $f \to f^*\mathcal{O}_Y(1)$. Intuitively, we can correspond $X_{s_i} = \{x \in X, \mathcal{L}_x = (s_i)_x \mathcal{O}_{X,x}\}$, and sent X_{s_i} to $Y_i = D_+(T_i)$.

  - This sort of says that projective spaces can be viewed as the classifying spaces for line bundles.

- An invertible sheaf $\mathcal{L}$ over a quasi-compact scheme X is ample if for every $\mathcal{F}$ quasi-coherent, there exists some n_0 such that $n \geq n_0$ implies that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

- Let $f : X \to \operatorname{Spec} A$, if X is Noetherian or f is separated, then there exists $m \geq 1$ such that $\mathcal{L}^{\otimes m}$ is very ample.

- $\mathcal{L}$ is ample if there exists $s_1, ..., s_r \in \mathcal{L}(X)$ such that X_{s_i} is affine for every i and that $X = \bigcup X_{s_i}$.

3 Homotopical Algebra

3.1 Simplicial Sets

- The realization functor is left adjoint to the singular functor. We have natural maps $X \to \operatorname{Sing}|X|$ and $|\operatorname{Sing} Y| \to Y$. Both of these are weak equivalences, where a map of simplicial sets is a weak equivalence if it is after applying realization.
- We define the mapping space $\underline{\mathcal{S}}(X, Y)$ to be the simplicial set with $\underline{\mathcal{S}}(X, Y)_n = \operatorname{Hom}(X \times \Delta^n, Y)$. A map $f : X \to Y$ is a simplicial homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity in the sense of 1-cells in the mapping space.
- We denote the category of simplicial Abelian groups by $\mathcal{A}$. It is also simplicial in the sense that $\underline{\mathcal{A}}(M, N)_n = \mathcal{A}(M \otimes \mathbb{Z}[\Delta^n], N)$, where $\mathbb{Z}[\Delta^n]$ is the free simplicial abelian group associated to the simplicial set.
- We define homotopy groups of simplicial sets/Abelian groups via $\pi_*(|X|)$, i.e., the homotopy groups of the realization.
- A simplicial Abelian group M gives rise to the Moore complex C_*, where $M_n \to M_{n-1}$ via the alternating sum $d_0 - d_1 + d_2 - ... \pm d_n$. This satisfies that $H_*(C_*(M)) = \pi_*(M)$. We can also define $H_*(X) = H_*(C_*\mathbb{Z}[X])$ for X a simplicial set in general.
- We can also define homology via the normalized chain complex, where $C_n^{\text{norm}}(M) = \bigcap_{i=0}^{n-1} \ker d_i : M_n \to M_{n-1}$ and the boundary map via d_n. C^{norm} gives an equivalence of categories between $\mathcal{A}$ and $\mathbf{Ch}^{\geq 0}$. The normalized complex is homotopy equivalent to the Moore complex (and hence gives the same homology groups).
- We define the suspension of $X \in \mathcal{S}$ via $S^1 \wedge X$. $S^n := S^1 \wedge S^{n-1}$. For a pointed set X, we define $\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[*]$. For a simplicial set X, its reduced homology is given as the homotopy groups of $\tilde{\mathbb{Z}}[X]$. Certain properties of $\tilde{\mathbb{Z}}$ guarantees the wedge axion, Künneth theorem, and excision.
  - In particular, $\tilde{\mathbb{Z}}[S^n]$ has the n-th homotopy group $\mathbb{Z}$ and anything else 0, and hence plays the role of the Eilenberg-Maclane spaces.
- Adjunction holds in the sense that $\underline{\mathcal{S}}_*(X \wedge Y, Z) \cong \underline{\mathcal{S}}_*(Y, \underline{\mathcal{S}}_*(X, Z))$.
- We define $\Omega X = \underline{\mathcal{S}}_*(S^1, \operatorname{Sing}|X|) \cong \operatorname{Sing} \Omega|X|$. Note that although $X \to \operatorname{Sing}|X|$ is a weak equivalence, $\underline{\mathcal{S}}_*(A, X) \to \underline{\mathcal{S}}_*(A, \operatorname{Sing}|X|)$ is not guaranteed to be one.
- Cohomology group of $X \in \mathcal{S}$ is given by $\tilde{H}^n(X) = \pi_0 \underline{\mathcal{S}}_*(X, \tilde{\mathbb{Z}}[S^n])$.

3.2 Spectra

- A spectrum is a sequence of simplicial sets E^* with maps $S^1 \wedge E^k \to E^{k+1}$. If the adjoint maps $E^k \to \Omega E^{k+1}$ are equivalences, we call the spectrum a Ω-spectrum.
- For example, $\mathbf{S} = (S^k)$ and $H\mathbb{Z} = (\tilde{\mathbb{Z}}[S^*])$.
- There is a pair of adjoint functors $R : \mathbf{Spt} \to \mathcal{S}_*$ via $E \to E^0$ and $\Sigma^\infty : \mathcal{S}_* \to \mathbf{Spt}$ via $\Sigma^\infty X \to (S^* \wedge X)$.
- Let E be a spectrum. The stable homotopy groups of E are defined as $\pi_q E = \operatorname{colim} \pi_{q+k} E^k$ (and $\pi_{-n} E = \operatorname{colim} \pi_k(\Omega^n E^k)$). A map of spectra is a stable equivalence if it induces isomorphisms on stable homotopy groups. Inverting all stable equivalence gives the stable homotopy category **HoSpt**.
- A spectrum gives a (co)homology theory by $E_n(X) = \pi_n(E \wedge X)$ and $E^n(X) = \pi_{-n}(E^X)$. The (co)homology group given by $H\mathbb{Z}$ is the same as the reduced (co)homology of X.

3.3 Model Structure

- One thing good about $\mathcal{S}$ is that every object behaves like a CW complex: if $K \subset L$, then there exists a functorial factorization $K = K(-1) \subset K(0) \subset ... \subset \bigcup K(n) = L$ where each $K(i-1) \subset K(i)$ is given by attaching cells:

$$\begin{array}{ccc} \coprod \partial\Delta^i & \longrightarrow & \coprod \Delta^i \\ \downarrow & & \downarrow \\ K(i-1) & \longrightarrow & K(i) \end{array}$$

- Trivial fibrations are maps that have the right lifting property with respect to $\partial\Delta^n \subset \Delta^n$, and fibrations are maps that have the right lifting property with respect to $\Lambda^n_i \subset \Delta^n$.

- Kan complexes are the fibrant objects. $\operatorname{Sing} Y$ is a Kan complex, and so is the nerve of a groupoid $\mathcal{G}$.

- Small object argument is used for many proofs. Essentially a map $\bigcup X_n \to Y$ comes from some map $X_n \to Y$.

- If $f : X \to Y$ is a (trivial) fibration and $A \subset B$, then so is $\underline{\mathcal{S}}(B, X) \to \underline{\mathcal{S}}(A, X) \times_{\underline{\mathcal{S}}(A,Y)} \underline{\mathcal{S}}(B, Y)$. The proof uses some tricks about adjunction. A weak equivalence between Kan complexes is a homotopy equivalence.

- For a Kan complex $X \in \mathcal{S}_*$, we define $\pi_q X = \pi_0\, \underline{\mathcal{S}}_*(S^q, X)$. We have $\pi_* X \cong \pi_*|X|$.

- We have a fibrant replacement functor Ex^∞, i.e., $X \subset \mathrm{Ex}^\infty X$ is a weak equivalence and $\mathrm{Ex}^\infty X$ is fibrant. WE can then define $\pi_q X = \pi_0\, \underline{\mathcal{S}}_*(S^q, \mathrm{Ex}^\infty X)$.

- (Proper Model Categories & Homotopy Pullback) $\mathcal{M}$ is left proper if in

  $$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

  where i a cofibration and f a weak equivalence, g is a weak equivalence. Dually we define right proper model categories. All the classical model category structures are proper. A commuting square in $\mathcal{M}$ is a homotopy pullback square

  $$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

  if for functorial factorization $B \to X \to D$ and $C \to Y \to D$ into trivial cofibrations followed by fibrations, $A \to X \times_D Y$ is a weak equivalence. $X \times_D Y$ is called the homotopy pullback.

  - If $\mathcal{M}$ is proper then the homotopy pullbacks are homotopy invariant. If the diagram is a pullback and $B \to D$ a fibration, the square is said to be homotopy cartesian.

- In **Spt**, we define a fibration (weak equivalence) to be a pointwise fibration (weak equivalence), and a cofibration to be those with the correct left lifting property. This gives a model structure.

- The stable structure on the other hand, define the weak equivalences to be the stable equivalences and fibrations to be those with the correct right lifting property. This is a cofibrantly generated model structure.

- Consider the category $\mathcal{S}^{\mathcal{C}}$. A natural transformation is a pointwise weak equivalence (fibration) if $X(c) \to Y(c)$ is a weak equivalence (fibration) for each $c \in \mathcal{C}$. This gives a model category structure on $\mathcal{S}^{\mathcal{C}}$ called the projective structure.
  - A cellular inclusion if a composite $X_0 \to ... \to \operatorname{colim} X_n$ where each $X_i \to X_{i+1}$ is the pushout along a disjoint union of maps $\partial\Delta^n \times \mathcal{C}(c, -) \to \Delta^n \times \mathcal{C}(c, -) \to$.
  - A cofibration is exactly a retract of a cellular inclusion.

3.4 Motivic Spaces

- The category of motivic spaces contains the pointed simplicial presheaves on $\mathcal{S}m/\mathcal{S}$. We define $\mathcal{M}_{\mathcal{S}} = \mathcal{S}_*^{(\mathcal{S}m/\mathcal{S})^{\mathrm{op}}}$. $\mathcal{M}_{\mathcal{S}}$ is given a model structure, the projective structure. It also has smash product $X \wedge Y(U) = X(U) \wedge Y(U)$ and a internal morphism object $\underline{\mathcal{M}_{\mathcal{S}}}$ right adjoint to the smash.
- $X \in \mathcal{M}_{\mathcal{S}}$ is locally fibrant if it is a Nisnevich sheaf up to homotopy. For $X \in \mathcal{M}_{\mathcal{S}}$ we define KX to be the functorial factorization $* \to KX \to X$ into a cofibration followed by a trivial fibration (in the projective structure). $X \to Y \in \mathcal{M}_{\mathcal{S}}$ is a local equivalence if for all locally fibrant Z the map $\underline{\mathcal{M}_{\mathcal{S}}}(KY, Z) \to \underline{\mathcal{M}_{\mathcal{S}}}(KX, Z)$ is a pointwise equivalence. Local cofibrations are the cofibrations in the projective structures, and local fibrations are the maps with the correct right lifting property.
  - This gives a model structure on $\mathcal{M}_{\mathcal{S}}$ with local weak equivalences detected pointwisely on locally fibrant objects.
- $X \in \mathcal{M}_{\mathcal{S}}$ is $\mathbb{A}^1$-fibrant if it is locally fibrant s.t. $X(\mathbb{A}^1 \times_S -) \to X(-)$ is a pointwise weak equivalence. An $\mathbb{A}^1$-equivalence is defined analogous to local equivalence. $\mathbb{A}^1$-cofibrations are the local cofibrations and $\mathbb{A}^1$-fibrations are those with the correct right lifting property.
  - This gives a model structure on $\mathcal{M}_{\mathcal{S}}$; its homotopy category is equivalent to the unstable homotopy category.
- We want motivic homology theories to be stable under smashing with the Tate object $T = S^1 \wedge \mathbb{G}_m$ and to commute with filtered colimits. We define $f\mathcal{M}_{\mathcal{S}} \subset \mathcal{M}_{\mathcal{S}}$ to be the category of finite spaces (sort of like CW complexes).
  - A functor $X : f\mathcal{M}_{\mathcal{S}} \to \mathcal{M}_{\mathcal{S}}$ is continuous if X induces maps $\underline{\mathcal{M}_{\mathcal{S}}}(v, w) \to \underline{\mathcal{M}_{\mathcal{S}}}(Xv, Xw)$. Such functors form a category $\mathcal{F}_{\mathcal{S}}$. Note that continuous functors give map $S^1 \wedge XS^n \to XS^{n+1}$ so (XS^n) give a spectrum.
  - We have a smash product in $\mathcal{F}_{\mathcal{S}}$, where $X \wedge Y \to Z$ is in correspondence with maps $Xu \wedge Yv \to Z(u \wedge v)$ that are natural in u and v.
  - The motivic sphere spectrum is the inclusion $\mathbf{S} : f\mathcal{M}_{\mathcal{S}} \to \mathcal{M}_{\mathcal{S}}$, and $\mathbf{S} \wedge X = X$. We can talk about $\mathbf{S}$-algebras.
- We have a pointwise structure on $\mathcal{F}_{\mathcal{S}}$ via the $\mathbb{A}^1$ structure on $\mathcal{M}_{\mathcal{S}}$. We define KX to be the functorial factorization $* \to FX \to X$ into a cofibration followed by a trivial fibration in the pointwise structure.
- $X \in \mathcal{F}_{\mathcal{S}}$ is ht-fibrant if it is pointwise fibrant such that for $\mathbb{A}^1$-equivalences $\phi : v \to w$ in $f\mathcal{M}_{\mathcal{S}}$, $Xv \to Xw$ is an $\mathbb{A}^1$-equivalence. An ht-equivalence is defined analogous to a local or an $\mathbb{A}^1$-equivalence in $\mathcal{M}_{\mathcal{S}}$. ht-cofibrations are the pointwise cofibrations and ht-fibrations are those with the correct right lifting structure.
  - ht-structure is a model category structure on $\mathcal{F}_{\mathcal{S}}$ that lifts to model structures on algebras and modules in $\mathcal{F}_{\mathcal{S}}$.

3.5 Motivic Spectra

- A T-spectrum (or motivic spectrum) is a sequence $(E^n) \subset \mathcal{M}_{\mathcal{S}}$ with $T \wedge E^n \to E^{n+1}$. These form a category $\mathbf{Spt}_S$ that comes with a model structure whose homotopy category is equivalent to the motivic stable homotopy category.
- Given X a spectrum in $\mathbf{Spt}$, we can define $(QX)^n = \operatorname{colim} \operatorname{Sing} \Omega^i |X^{n+i}|$, this is a Ω-spectrum with a canonical map $X \to QX$. Now in $\mathbf{Spt}_S$, we declare a map $X \to Y$ to be a stable equivalence if $(QX)^n \to (QY)^n$ is an $\mathbb{A}^1$-equivalence for each n. Similarly we define stable fibrations and hence a stable structure on $\mathbf{Spt}_S$.
- Let $T^n = T^1 \wedge T^{n-1}$ and $T^0 = S^0$. For $X \in \mathcal{F}_{\mathcal{S}}$, we define $evX \in \mathbf{Spt}_S$ to be the spectrum $(X(T^n))$. This gives a functor $\mathcal{F}_{\mathcal{S}} \to \mathbf{Spt}_S$ that preserves limits and colimits. This forces the stable structure on $\mathcal{F}_{\mathcal{S}}$ (after functorial replacement by homotopy functors (i.e., those sending weak equivalences to equivalences, or the fibrant objects in the ht-structure) via the ht-structure).
- There is a left adjoint $F : \mathbf{Spt}_S \to \mathcal{F}_{\mathcal{S}}$ and this is a Quillen equivalence after putting the stable structure on $\mathcal{F}_{\mathcal{S}}$. The stable model structure induces model category structure on algebras and modules over $\mathcal{F}_{\mathcal{S}}$.

4 Geometric Invariant Theory & Moduli

- One goal was to construct good quotient: for example, $\mathbb{C}^\times$ acts on $\mathbb{C}$, but the fiber of the map $\mathbb{C} \to \mathbb{C}/\mathbb{C}^\times$ are not closed. We can form categorical quotient for affine schemes, by letting $\operatorname{Spec} A/G = \operatorname{Spec} A^G$, but it is not always the "good" geometric quotient.
- (Geometric Quotient). Q is a geometric quotient if there exists $\pi : X \to Q$ such that π surjective with fibers G-orbits, $\pi^{-1}(U)$ open iff U open, and that $\Gamma(U, \mathcal{O}_U) = \Gamma(\pi^{-1}(U), \mathcal{O}_{\pi^{-1}(U)})^G$.
- Good categorical quotients exist for affine varieties, given furthermore that G is reductive. However, it is not always desired.
  - For example, $\mathbb{C}^\times$ acts on $\operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1$ by sending $\lambda \cdot f(x)$ to $f(\lambda x)$. Now, $\mathbb{A}^1 // \mathbb{C}^\times = \operatorname{Spec} \mathbb{C}$ consists of only one point, while there are two orbits.
  - Let $\mathbb{C}^\times$ acts on $\mathbb{A}^2$ by sending $t \cdot (x, y)$ to (tx, ty). Then $(\mathbb{A}^2)^{\mathbb{C}^\times} = \mathbb{C}$, and $\mathbb{C}^2 // \mathbb{C}^\times = \operatorname{Spec} \mathbb{C}$ has only one point, while the orbits are $\{(0,0)\}$, $\{\mathbb{C}^\times \times 0\}$, $\{0 \times \mathbb{C}^\times\}$, and (complement of 0 in) $\operatorname{Spec} \mathbb{C}[xy^{-1}] = \mathbb{A}^1$. In short, the dimension is not right. Note that the closure of the orbits contain $(0,0)$.
  - On the other hand, let Let $\mathbb{C}^\times$ acts on $\mathbb{A}^2$ by sending $t \cdot (x, y)$ to $(tx, t^{-1}y)$, then the orbits are $\{(0,0)\}$, $\{\mathbb{C}^\times \times 0\}$, $\{0 \times \mathbb{C}^\times\}$, and (complement of 0 in) $\operatorname{Spec} \mathbb{C}[xy] = \mathbb{A}^1$. The orbits in the last one are closed.
  - In fact, points in $X//G$ corresponds to closed orbits.
- Assume G reductive. A point x is stable if G_x is finite and $G \cdot x$ is closed. The set of stable points X^s is open, and $X^s \to X^s//G$ is a geometric quotient. In addition, if $X^s \neq \varnothing$, then $\dim X - \dim G = \dim X//G$.
- One thing not so good is that $U//G \to X//G$ is not always an open embedding even if $U \to X$ is. For example, consider $U = \mathbb{A}^2 \backslash \{(0,0)\}$, then $U//G = \mathbb{P}^1$ while $X//G$ is a point. (We can glue things up to form $U//G$).
- For $X \subset \mathbb{P}^m$ projective variety, one can show that X^{ss} is covered by G-invariant affines where $x \in X^{ss}$ iff $x \in X_f$ for some G-invariant f. Note that in general we can't just glue things up, because it is not necessarily true that X can be covered by G-invariant open affines.

- We then form $X//G = X^{ss}//G$, where the latter comes by gluing. $X \to X^{ss}//G$ is only rational and defined on semi-stable points.
  - This is a categorical quotient and almost geometric.
  - $X//G$ is a projective variety.
  - In general $X^s/G \subset X//G$ and $\dim X//G = \dim X - \dim G$.
- Let $V \subset \mathbb{P}^n$ be a smooth hypersurface of degree d, we can view it as a point in $\mathbb{P}^{N_d-1}$ where $N_d = \binom{n+d}{d}$. $[V]$ is semi-stable but not stable w.r.t. the $\mathbb{P}GL(n+1)$ action.
  - A quadric over $\mathbb{C}$ in $\mathbb{P}^m$ is classified by the rank. In other words, $Q \simeq_{\mathbb{P}GL(m+1)} V(x_0^2 + ...x_k^2)$ for some unique k. There are hence $m+1$ orbits of the action of $\mathbb{P}GL(m+1)$.
  - One can show, with a 1-parameter subgroup argument, that the closure of the orbit $G \cdot (x_0^2 + ... + x_k^2)$ contains $G \cdot (x_0^2 + ... + x_{k-1}^2)$. Also, $G \cdot x_0^2$ is not semistable.
  - In fact, $\mathbb{P}^{N_2}//\mathbb{P}GL(m+1)$ consists of one point, and there is a unique polystable orbit: the orbit of smooth quadrics. This orbit is open in $X = \mathbb{P}^{N_2}$ but closed in X^{ss}.

5 Equivariant Homotopy Theory

We fix G a finite group or a compact Lie group (or perhaps more generally, a compact topological group).

5.1 Unstable Theory

- A G-space X is defined via a G-action and expected compatibility axioms. G-equivariant maps are also defined as expected. We write the category of G-spaces as G**Top**, and there are two choices for what the morphisms are: we can use the G-equivariant maps between two G-spaces, or we can use the G-space of all maps between the underlying spaces with G acting via conjugations, which allows G**Top** to be enriched over itself. We denote the former as $\mathrm{Map}^G(X, Y)$ and the latter as $G\mathbf{Top}(X, Y)$. Note that $(G\mathbf{Top}(X, Y))^G = \mathrm{Map}^G(X, Y)$.
- Let X a G-set, X^H is the fixed points under a subgroup action, and G_x is the isotropy group, the elements in G that fixes x. Note that $X^H \simeq \mathrm{Map}(G/H, X)$.
- Let $G \to H$ a group map. We naturally have a functor $H\mathbf{Top} \to G\mathbf{Top}$. This map has left and right adjoints, which can be understood as the Kan extensions along $BG \to BH$, where BG and BH are the one-point groupoid with automorphism set G and H.
- Let V be a finite-dimensional real representation of G (i.e., an inner product space on which G acts in a way compatible with the inner product). The one-point compactification of V is denoted S^V, and this is exactly $D(V)/S(V)$. S^V is called a representation sphere.
- A G-homotopy is a map $X \times I \to Y$ between G-spaces where G acts trivially on I.
- We define G-CW complexes by gluing cells, where the cells are $G/H \times D^n$ and $G/H \times S^n$ with trivial actions where H is any subgroup of G.
  - Note that $[G/H \times S^n, X] \simeq \pi_n(X^H) =: \pi_n^H(X)$. We define a weak-equivalence to be a map such that $\pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism for all H.
  - The spirit is that $G/H \times D^n$ represents a cell on which H acts trivially. For example, consider S^1 acts on $\mathbb{R}^2$ by rotation. Let $V = \mathbb{R}^2$ with this action. S^V is a G-CW complex with two fixed points $S^1/S^1 \times *$ and one 1-cell $S^1 \times I$ attached to the endpoints (rotate the geodesic between two endpoints via the S^1-action).

- Let θ : {conjugacy classes of subgroups of G} $\to \{x \in \mathbb{Z}, x \geq -1\}$. We say a map is θ-connected if f^H is $\theta(H)$-connected for every H, and a G-CW complex is θ-dimensional if all cells of orbit type G/H have dimension at most $\theta(H)$. The key point is the equivariant Whitehead theorem: a weak equivalence between G-CW complexes is a G-homotopy equivalence.
  - In fact we can define a model structure on G**Top**, where fibrations or weak equivalences are those maps f such that f^H is a fibration or a weak equivalence for every subgroup H.

5.2 Stable Theory

- For motivation, we consider the $\mathcal{S}$-category, in which the objects are topological spaces and morphisms are $\mathrm{Map}(X, Y) = \mathrm{colim}\, \mathrm{Map}(\Sigma^n X, \Sigma^n Y)$ where the latter is in **Top**.
- (Equivariant Whitney's Theorem). A G-manifold is a manifold M with smooth G-action. Let M compact G-manifold, then there exists a G-equivariant embedding $M \to V$ where V is some finite-dimensional real G-representation.
  - This leads to equivariant version of everything, PT-construction, Atiyah duality, etc. However, we first need to define the right $\mathcal{S}$-category equivariantly, where $\mathrm{Map}(X, Y) := \mathrm{colim}_V \mathrm{Map}(\Sigma^V X, \Sigma^V Y)$ where V takes place over all finite-dimensional real representations V.
- (Orbit Category). Earlier we defined the orbit category $\mathcal{O}_G$ in which the objects are G/H for H some closed subgroup, and morphisms G-equivariant maps. For the equivariant $\mathcal{S}$-category, we need more morphisms in the orbit category called the transfer maps.
  - Suppose M embeds in some G-representation V, then the PT construction gives a map $S^V \to T\nu \to T(\nu \oplus \tau) \to S^V \wedge M_+$. Suppose $K \subset H$, we get maps $S^V \to S^V \wedge (H/K)+$, and inducing this gives maps $(G/H)_+ \wedge S^V \to (G/K)_+ \wedge S^V$, which are the extra maps we need in $\mathcal{O}_G$.
  - Now we have two functors from $\mathcal{O}_G$ to the equivariant $\mathcal{S}$-category. One of them is covariant and sends a map to its representative in the colimit, while the other one is contravariant via these transfer maps.
- (Burnside Category and Mackey Functor). The Burnside category B_G is the full subcategory of the $\mathcal{S}$-category spanned by orbits G/H's. The maps in B_G is a compositions of two maps in the image of the two functors depicted above. It is enriched in **Top**. The algebraic Burnside category is the homotopy category $\pi_0 B_G$ and is enriched in **Ab**. A Mackey functor is a **Ab**-enriched functor $\pi_0 B_G \to$ **Ab**.

6 Higher Topos Theory

6.1 Introduction

- (The homotopy type of) A topological space is encoding essentially the same data as an ∞-groupoid does. Therefore, we can define an ∞-category to be a category enriched over **Top**.
- On the other hand, we can define an ∞-category via inner horn lifting property. $\mathrm{Sing}\, X$ and $N\mathcal{C}$ are ∞-categories, and the former is a Kan complex.
  - We do not require the uniqueness of the lift, because we only need the lift to be unique up to higher homotopy. Consider the fundamental groupoid of a space for example.

- We can view the homotopy category of a topological category as a category enriched over $\mathcal{H}$, the homotopy category of CW complexes, and two topological category should be seen as weak equivalent if the induced functor on $\mathcal{H}$-enriched categories is an equivalence.
  - However, note that $h\mathcal{C}$ alone does not determine $\mathcal{C}$ up to equivalence; we need an equivalence $h\mathcal{C} \to h\mathcal{D}$ to say $\mathcal{C} \simeq D$.
- To connect the two notions, we use simplicial categories. Note that one can go between a simplicial category and a topological category via Sing and $|-|$. One can also talk about the $\mathcal{H}$-enriched category $h\mathcal{C}$ for a simplicial category $\mathcal{C}$.
  - We define the simplicial nerve of a simplicial category $N\mathcal{C}$ via $\mathbf{sSet}(\Delta^n, N\mathcal{C}) = \mathbf{sCat}(\mathbb{C}[\Delta^n], \mathcal{C})$ where $\mathbb{C}[-]$ sends an ordered set to a simplicial category. We can extend $\mathbb{C}$ to $\mathbb{C} : \mathbf{sSet} \to \mathbf{sCat}$, and it is left adjoint to N.
  - Let $\mathcal{C}$ simplicial category. If $\mathrm{Map}(X, Y)$ is a Kan complex for every pair (X, Y), then $N\mathcal{C}$ is an ∞-category. In particular, if $\mathcal{C}$ is a topological category, then $N\mathcal{C} := N\,\mathrm{Sing}\,\mathcal{C}$ is an ∞-category.
  - In fact one can show that N and $\mathbb{C}[-]$ are homotopy inverse to each other:

    $$|\mathrm{Map}_{\mathbb{C}[N\mathcal{C}]}(X, Y)| \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

    is a weak homotopy equivalence of topological spaces, where $X, Y \in \mathcal{C}$ a topological category.
- We define $\mathcal{C}^{\mathrm{op}}$ by inverting the mapping spaces.
- Let $\mathcal{C}$ ∞-category, we can define $\mathrm{Hom}^R_{\mathcal{C}}(x, y)$ with n-simplices being the collection of $z : \Delta^{n+1} \to \mathcal{C}$ such that $z|_{\Delta^{\{n+1\}}} = y$ and $z|_{\Delta^{\{0,1,\ldots,n\}}} = x$. We can similarly define $\mathrm{Hom}^L_{\mathcal{C}}(x, y)$ and this is homotopy equivalent to $\mathrm{Hom}^R_{\mathcal{C}}(x, y)$.
  - $\mathrm{Hom}^R_{\mathcal{C}}(x, y)$ is a Kan complex and represents $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathcal{H}$. $\mathcal{C}$ being an ∞-category allows compositions of maps.
- $h : \mathbf{sSet} \to \mathbf{Cat}$ is the left adjoint to N.
  - For $\mathcal{C}$ an ∞-category, we can define $\pi(\mathcal{C})$ that is naturally equivalent to $h\mathcal{C}$, with a more concrete construction: objects are objects in $\mathcal{C}$ and morphisms are homotopy classes of edges $\Delta^1 \to \mathcal{C}$ that goes between x and $y \in \mathcal{C}$ (two edges being homotopic is an equivalence relation on edges joining x and y, and compositions are well-defined in $\pi(\mathcal{C})$.
  - A morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$ is an equivalence if it is an isomorphism in $h\mathcal{C}$.
- The inclusion of Kan complexes into ∞-categories has a right adjoint: $\mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}'$ is the largest Kan complex contained in $\mathcal{C}$ (every edge in $\mathcal{C}'$ is an equivalence).
- Working on $h\mathcal{C}$ is somewhat similar to working on $\mathcal{C}$, but one distinction is that the homotopy between two maps in $\mathcal{C}$ should be a datum. Commutative diagrams in $h\mathcal{C}$ corresponds to homotopy commutative diagrams in $\mathcal{C}$ (for example, a 1-functor $\mathcal{C} \to \mathcal{H}$ does not always lift to $\mathcal{C} \to \mathbf{Top}$).
  - This motivates the definition of a homotopy coherent diagram in an ∞-category $\mathcal{C}$, namely it is a functor $\mathcal{J} \to h\mathcal{C}$ with additional data that allows us to lift the functor to $\mathcal{J} \to \mathcal{C}$. We can also define a homotopy coherent diagram to be a map of simplicial sets $N\mathcal{J} \to \mathcal{C}$.
- The general slogan: if two objects in $\mathcal{C}$ are equivalent, then the ∞-categorical notions should be the same for them.

– Let $\mathcal{C}$ be an ∞-category, then $\mathrm{Fun}(K, \mathcal{C}) := \mathbf{sSet}(K, \mathcal{C})$. This in particular gives the notion of the ∞-category of functors between ∞-categories.

– One can define the join of ∞-categories such that $N(\mathcal{C} \star \mathcal{C}') = N\mathcal{C} \star N\mathcal{C}'$. In particular, the join of ∞-categories is an ∞-category.

  * Let $K \in \mathbf{sSet}$, the left/right cones $K^{\triangleleft}$ and $K^{\triangleright}$ are defined as $\Delta^0 \star K$ and $K \star \Delta^0$. Both have a distinguished vertex.

– $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective if hF is, and is fully faithful if $\mathrm{Map}_{h\mathcal{C}}(X, Y) \to \mathrm{Map}_{h\mathcal{D}}(FX, FY)$ is an isomorphism in $\mathcal{H}$.

– A subcategory of $\mathcal{C}$ is a pullback after choosing a subcategory $h\mathcal{C}' \subset h\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ Nh\mathcal{C}' & \longrightarrow & Nh\mathcal{C} \end{array}$$

– Let $\mathcal{C}$ ∞-category, defining the initial/final objects to be the ones in $\mathcal{C}$ as a 1-category would be too strict. Instead, we say $Y \in \mathcal{C}$ is final if $\mathrm{Hom}^R_{\mathcal{C}}(X, Y)$ is contractible.

– The above notions allow us to define colimit/limit for $p : K \to \mathcal{C}$ a map between simplicial sets as the initial/final object in diagrams $K \to \mathcal{C}$ under/below p. On the other hand one can also identify such under/overcategories with the category of maps $K^{\triangleleft}$ or $K^{\triangleright} \to \mathcal{C}$.

• In particular, we define $N\mathbf{Kan} = \mathfrak{S}$ to be the ∞-category of spaces, where $\mathbf{Kan} \subset \mathbf{sSet}$ is simplicial.

6.2 Fibrations of Simplicial Sets

• $X \to S$ is a Kan (trivial) fibration if it has RLP for all $\Lambda^n_k \to \Delta^n$ ($\partial\Delta^n \to \Delta^n$). $A \to B$ is anodyne (cofibration) if it has LLP for all Kan (trivial) fibration.

  – We can define left/right/inner fibration/anodyne by restricting k (e.g., an ∞-category is characterized by $\mathcal{C} \to *$ inner fibration).

  – The small object argument shows that any map $X \to Z$ can be factorized into a cofibration/anodyne followed by trivial/Kan fibration (perhaps with left/right/inner).

• (Cofibered in Groupoids over $\mathcal{D}$). Let $F : \mathcal{D} \to \mathbf{Gpd}$ a functor. We can construct $\mathcal{C}_F$ with a functor $\mathcal{C}_F \to \mathcal{D}$. However, not all $\mathcal{C} \to \mathcal{D}$ arise from a functor to $\mathbf{Gpd}$, so we need the notion of $\mathcal{C}$ being cofibered in groupoids over $\mathcal{D}$, and informally functors $F : \mathcal{D} \to \mathbf{Gpd}$ and categories cofibered in groupoids over $\mathcal{D}$ are equivalent.

  – In particular, $F : \mathcal{C} \to \mathcal{D}$ is cofibered in groupoids over $\mathcal{D}$ iff NF is a left fibration of simplicial sets.

  – Precisely, we can construct a Quillen equivalence $\mathbf{sSet}^{\mathbb{C}[S]} \to \mathbf{sSet}_{/S}$ where fibrants are the left fibrations over S. This suitable model structure on $\mathbf{sSet}_{/S}$ is called the covariant model structure.

• Kan complexes are those $X \to *$ with RLP w.r.t. left fibration, so for any left fibration $p : X \to S$, the fiber $X_s = X \times_S \{s\}$ is a Kan complex (an ∞-groupoid).

  – In particular, we can define $f_! : X_s \to X_{s'}$ given an edge $s \to s'$ with higher coherence data. So a left fibration over S is more or less like a functor $S \to \mathfrak{S}$.

  – In fact, $p : X \to S$ is a Kan fibration iff $X_s \to X_{s'}$ is an isomorphism in $\mathcal{H}$. This gives a construction $X_{s'} \to X_s$.

- A functor $\mathcal{C} \to \mathfrak{S}$ is a "cosheaf of spaces" on $\mathcal{C}$. Examples of such include the functors sending $D \to \mathrm{Map}_{\mathcal{C}}(C, D)$, and such functors arise from a left fibration $f : \mathcal{C}_{C/} \to \mathcal{C}$. In fact, $\mathcal{C}_{p/} \to \mathcal{C}$ is a left fibration for $p : K \to \mathcal{C}$ a diagram.

- (Straightening & Unstraightening). Let S simplicial set and $\mathcal{C}$ simplicial category. $\phi : \mathbb{C}[S] \to \mathcal{C}^{\mathrm{op}}$ a simplicial functor. With the correct model structure, we have a Quillen adjunction:

$$\mathbf{sSet}_{/S} \underset{\mathrm{Un}_\phi}{\overset{\mathrm{St}_\phi}{\rightleftarrows}} \mathbf{sSet}^{\mathcal{C}}$$

  and this is a Quillen equivalence if ϕ is an equivalence of simplicial categories. When ϕ is the identity, we call the maps St_S and Un_S.

  - The general philosophy: we can define straightening $\mathrm{St} : \mathbf{sSet}_{/S} \to \mathcal{M}^{\mathbb{C}[S]}$, where the target has a matching model structure as on $\mathbf{sSet}_{/S}$ via the Quillen inverse Un. For an arbitrary $\mathcal{M}$, we cannot expect to always being able to pull the model structure back.

- Let $\mathcal{C}$ a category with small colimit, then a cosimplicial object $C^* \in \mathcal{C}$ determines a functor $\mathbf{sSet} \to \mathcal{C}$ via the coend construction. We denote $FS = |S|_{C^*}$. F has a right adjoint Sing_{C^*}.

  - In particular, suppose $X = \{x\} \in \mathbf{sSet}$. We can identify St_X with the geometric realization $|-|_{Q^*}$ for some Q^* cosimplicial in $\mathbf{sSet}$.
  - One can show that $|-|_{Q^*}$ and Sing_{Q^*} give a Quillen equivalence for $\mathbf{sSet}$ and itself, with the correct model structure.

- (Joyal Model Structure). We now have a model structure on $\mathbf{sSet}$ where cofibrations are inclusions, fibrants are ∞-categories, and weak equivalences are those that are equivalence after applying $\mathbb{C}[-]$ and move to $\mathbf{sCat}$. Also, $(\mathbb{C}, N)$ gives a Quillen equivalence between $\mathbf{sSet}$ (with this model structure) and $\mathbf{sCat}$.

- $X \to S$ being an inner fibration implies that each fiber is an ∞-category. However the fibers do not really functorially depend on S.

- A correspondence between two categories is a functor $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}' \to \mathbf{Set}$. We can form $\mathcal{C} \star^{\mathcal{M}} \mathcal{C}'$. Given $F : \mathcal{C}' \to \mathcal{C}$, we can define $M_F(c, c') = \mathcal{C}(c, Fc')$.

  - Given an inner fibration $X \to S$ and $f : s \to s'$ in S an edge, we can define a correspondence between X_s and $X_{s'}$.
  - We might want to view an inner fibration as a functor from S to the ∞-category $\infty\mathbf{Cat}$ where morphisms in the latter are correspondences. However, this does not work because correspondences do not compose nicely. The correct notion would be Cartesian fibrations.

- (Minimal Kan Complexes). We can classify Kan complexes via minimal Kan complexes; an equivalence between the latter is an isomorphism. In fact, a more general notion, minimal ∞-categories, is possible.

- (n-Truncated). We can define n-categories essentially by saying two maps $\Delta^m \to \mathcal{C}$ are the same if they agree on the boundary (where $m \geq n$.)

  - This leads to the notion of k-truncated: explicitly, a Kan complex K is k-truncated if $\pi_i(X, x) \simeq *$ for $k < i$ and $x \in X$. All the mapping spaces of $\mathcal{C}$ are $(k-1)$-truncated iff the minimal model of $\mathcal{C}$ is an k-category.

- We want to know whether a correspondence M arises from such a functor. This requires the notion of Cartesian morphisms, and motivates the following definition:

  - Let $p : X \to S$ inner fibration (RLP for Λ^n_k where $0 < k < n$). $f : x \to y$ an edge in X is called p-Cartesian if $X_{/f} \to X_{/y} \times_{S_{/py}} S_{/pf}$ is a trivial Kan fibration.

  - Let $p : N(\mathcal{M}) \to \Delta^1$ a map, then $f : x \to y$ in $\mathcal{M}$ which maps isomorphically onto Δ^1 is p-Cartesian iff it is Cartesian in the classical sense.
  - In particular, let $p : \mathcal{C} \to \mathcal{D}$ an inner fibration between ∞-categories and $f : c \to c'$ morphism in $\mathcal{C}$. f is an equivalence iff it is p-Cartesian and pf is an equivalence.

- Let $p : X \to S$ between simplicial sets. p is a Cartesian fibration if it is an inner fibration, and that for every edge $f : x \to y$ in S and $\tilde{y} \in X$ lifting y, there exists $\tilde{f} : \tilde{x} \to \tilde{y}$ p-Cartesian that lifts f.
  - Now let $p : X \to S$ an inner fibration, we can associate an ∞-category X_s to each $s \in S$, and a correspondence $X_e : X \times_S \Delta^1 \to \Delta^1$ where $(X_e)_0 = X_s$ and $(X_e)_1 = X_{s'}$.
  - Being a Cartesian fibration is essentially saying that the correspondences come from functors $X_{s'} \to X_s$. Or in other word a Cartesian fibration gives a functor $S^{\text{op}} \to \infty\mathbf{Cat}$. However, it should be noted that equivalence between Cartesian fibrations do not give equivalence between functors. This motivates the definition of marked simplicial sets.
  - An inner fibration is a right fibration iff it is Cartesian and that every fiber is a Kan complex, or if every edge in X is p-Cartesian.

- We can always talk about locally p-Cartesian or locally Cartesian fibration by pulling back to $X \times_S \Delta^1 \to \Delta^1$.
  - A locally Cartesian fibration is a Cartesian fibration iff every locally p-Cartesian edge in X is p-Cartesian.

- Let $p : \mathcal{C} \to \mathcal{D}$ (co)Cartesian fibration. $q : K \to \mathcal{C}$ diagram. Then $\mathcal{C}_{/q} \to \mathcal{D}_{/pq}$ is a (co)Cartesian fibration, and an edge in $\mathcal{C}_{/q}$ is p'-(co)Cartesian iff the image of f in $\mathcal{C}$ is.

- Let $p : \mathcal{C} \to \mathcal{D}$ an inner fibration of ∞-categories, $\bar{e} : pX \to pY$ a morphism in $\mathcal{D}$, $e : X' \to Y$ locally p-Cartesian lifting $\bar{e}$. Then we have a fiber sequence in $\mathcal{H}$: $\text{Map}_{\mathcal{C}_{pX}}(X, X') \to \text{Map}_{\mathcal{C}}(X, Y) \to \text{Map}_{\mathcal{D}}(pX, pY)$.

- Let $p : \mathcal{C} \to \mathcal{D}$ inner fibration of ∞-cats. Let $f : Y \to Z$ in $\mathcal{C}$. f is p-Cartesian iff for every $X \in \mathcal{C}$, the following diagram is homotopy Cartesian:

$$\begin{array}{ccc} \text{Map}(X,Y) & \longrightarrow & \text{Map}(X,Z) \\ \downarrow & & \downarrow \\ \text{Map}(pX,pY) & \longrightarrow & \text{Map}(pX,pZ) \end{array}$$

6.3 The ∞-Category of ∞-Categories

- We define $\infty\mathbf{Cat}$ to be the ∞-category $N(\infty\mathbf{Cat}^{\Delta})$ where $\infty\mathbf{Cat}^{\Delta}$ is the simplicial category with mapping spaces defined by the taking all invertible functors between two ∞-categories.

- We want to do fibrant replacements: let $p : X \to S$, we can factor it through $q \circ \phi : X \to Y \to S$ where q is a Cartesian fibration. We want to know what edges in X has image q-Cartesian in Y, and this requires the notion of marked simplicial sets.
  - Motivation: with right fibrations we can induce functors from the category of right fibrations over S to $\text{Fun}(S^{\text{op}}, \mathbf{Space})$. However, this does not work when we go to Cartesian fibrations, because two Cartesian fibrations can be isomorphic over S with completely different collections of Cartesian edges. We therefore need to not only remember the fibration but also the collection of Cartesian edges.
  - In essence, a marked simplicial set $(X, \mathcal{E})$ consists of $X \in \mathbf{sSet}$ and $\mathcal{E}$ a family of edges in X that contains all degenerate edges. We denote the category $\mathbf{sSet}^+$. We want to endow each ${\mathbf{sSet}^+}_{/S}$ a model structure.

- We define a notion called marked anodyne maps in $\mathbf{sSet}^+$ so that $p : X \to S$ has RLP w.r.t. all marked anodyne maps iff p is an inner fibration, edge e marked iff $p(e)$ marked and e p-Cartesian, and that every marked edge in S can be lifted to an marked edge in X.
- Two default marked simplicial sets: $S^\flat$ where marked edges are the degenerates, and $S^\natural$ where every edge is marked.

- $X^\sharp$ is used to denote, given $p : X \to S$, the marked simplicial set $(X, \mathcal{E})$ where $\mathcal{E}$ contains all p-Cartesian edges. Now, $Y \to S^\natural$ has RLP w.r.t. marked anodyne maps iff $Y \to S$ is Cartesian and $Y = Y^\flat$.

- We can now define a model structure on $\mathbf{sSet}^+{}_{/S}$ where cofibrations are those that are cofibrations in $\mathbf{sSet}$ and weak equivalences are the Cartesian equivalences.

  - Now, X is fibrant iff $X \simeq Y^\natural$, where $Y \to S$ is a Cartesian fibration.

- (Fibered Categories). Let $\chi : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a functor. The Grothendieck construction assigns a new category $\tilde{\mathcal{C}}$ where objects are $(C, \eta), C \in \mathcal{C}$, $\eta \in \chi(C)$, and morphisms are $(f, \alpha), f : C \to C'$, $\alpha : \eta \to \chi(f)(\eta')$. There is an equivalence between functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and categories fibered over $\mathcal{C}$ (given $\phi : C \to C'$ and F' over C', we can lift ϕ to $\mathcal{F}$).

  - We try to mimic the approach by replacing $\mathcal{C}$ by a simplicial set S, $\mathbf{Cat}$ by $\infty\mathbf{Cat}$, and fibered categories by Cartesian fibrations. Note that the category of simplicial functors $\mathbb{C}[S]^{\mathrm{op}} \to \mathbf{sSet}^+$equipped with the projective model structure has the underlying ∞-category equivalent to $\mathrm{Fun}(S^{\mathrm{op}}, \infty\mathbf{Cat})$.
  - (Straightening-Unstraightening). Let $\phi : \mathbb{C}[S] \to \mathcal{C}^{\mathrm{op}}$ simplicial functor, we have an adjunction:

$$\mathbf{sSet}^+{}_{/S} \underset{\mathrm{Un}^+_\phi}{\overset{\mathrm{St}^+_\phi}{\rightleftarrows}} (\mathbf{sSet}^+)^{\mathcal{C}}$$

    and this is a Quillen adjunction where on the LHS we have the Cartesian model structure and on the RHS we have the projective model structure. This is a Quillen equivalence if ϕ is an equivalence, which in particular gives our description above.
  - Recall that given a simplicial model category, we can first restrict to the fibrant-cofibrant objects (and the subcategory is Kan-enriched), and then apply the coherent nerve functor. This gives an underlying ∞-category.

- Let f be a map between Cartesian fibrations over S as marked simplicial sets. f is a categorical equivalence iff fiberwise f induces a categorical equivalence, or iff $X^\natural \to Y^\natural$ is a Cartesian equivalence in $\mathbf{sSet}^+_{/S}$.

- $p : X \to S$ is a categorical fibration if it is a Cartesian fibration, and the converse is true if S is Kan.

- (Universal Fibration). We have a natural inclusion $\infty\mathbf{Cat}^\Delta \to \mathbf{sSet}^+$ which can be seen as a fibrant object $\mathcal{F} \in (\mathbf{sSet}^+)^{\infty\mathbf{Cat}^\Delta}$. Applying $\mathrm{Un}^+_{\infty\mathbf{Cat}^{\mathrm{op}}}$, we identify it with a Cartesian fibration $q : Z \to \infty\mathbf{Cat}^{\mathrm{op}}$ as an object in $\mathbf{sSet}^+_{/\infty\mathbf{Cat}^{\mathrm{op}}}$, and this is called the universal Cartesian fibration. For every $\mathcal{C} \in \infty\mathbf{Cat}$, its fiber of q $Z \times_{\infty\mathbf{Cat}^{\mathrm{op}}} \{\mathcal{C}\}$ is canonically equivalent to $\mathcal{C}$.

  - In fact, every Cartesian fibration between small simplicial sets $X \to S$ is classified by a map $\phi : S \to \infty\mathbf{Cat}^{\mathrm{op}}$.

- (Limit of a Diagram). Let $p : K \to \infty\mathbf{Cat}^{\mathrm{op}}$ a diagram classified by a Cartesian fibration $p : X \to K$, then $\lim p \simeq \mathrm{Map}^\flat_K(K^\sharp, X^\natural)$ in $h\infty\mathbf{Cat}$. Similarly, let $p : K \to \mathbf{Space}$ classified by $X \to K$ left fibration, then there is a natural isomorphism $\lim p \simeq \mathrm{Map}_K(K, X)$ in $\mathcal{H}$.

- (Colimit of a Diagram). Let $p : K^{\mathrm{op}} \to \infty\mathbf{Cat}$ a diagram classifying a Cartesian fibration $X \to K$, then $\operatorname{colim} p \simeq X^{\sharp}$ in $h\infty\mathbf{Cat}$. Similarly, let $p : X \to \mathbf{Space}$ classifying a left fibration $X \to K$, then $\operatorname{colim} p \simeq X$ in $\mathcal{H}$.

6.4 Limits and Colimits

- (Cofinal Maps). We use cofinal maps to compute colimit after replacing the diagram with a smaller one. The explicit definition: $p : S \to T$ is cofinal if for any $X \to T$ right fibration, $\operatorname{Map}_T(T, X) \to \operatorname{Map}_T(S, X)$ is a homotopy equivalence.
  - Let $K' \to K$ cofinal and $K \to \mathcal{C}$, where $\mathcal{C}$ an ∞-category. There is a homotopy equivalences of Kan complexes between fibers of $\mathcal{C}_{p/} \to \mathcal{C}$ and $\mathcal{C}_{p \circ v/} \to \mathcal{C}$.
  - In particular, if $K^{\triangleright} \to \mathcal{C}$ is the colimit of $K \to \mathcal{C}$, then the restriction to $K'^{\triangleright}$ is the colimit of $K' \to \mathcal{C}$. The converse tells that $K' \to K$ is cofinal.
  - Cofinal maps are weak equivalences. $i : A \to B$ is cofinal iff it is a contravariant equivalence in $\mathbf{sSet}_{/B}$. In particular, if B Kan, then i is cofinal iff i is a weak homotopy equivalence.

- (Smooth Maps). Let $p : X \to Y$ between simplicial sets. In general p induces a Quillen adjunction $(p_!, p^*)$ between the contravariant model categories $\mathbf{sSet}_{/X}$ and $\mathbf{sSet}_{/Y}$. p^* has a right adjoint p_*, but (p^*, p_*) is not necessarily a Quillen adjunction.
  - p is a smooth map if for every p' pulled back from p, (p'^*, p'_*) is a Quillen adjunction. This happens iff p'^* preserves contravariant equivalences.
  - Now, suppose we have a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow[f]{} & S \end{array}$$

    with f cofinal and p smooth, then f' is cofinal.

- Let $f : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ an ∞-category. f is cofinal iff for every $D \in \mathcal{D}$, $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible. This in particular leads to Quillen's Theorem A in his work *Higher K-Theory.*

- For every $p : X \to \mathcal{C}$, we can find $\mathcal{C}^{p/}$ by the adjoint functor theorem, such that $\operatorname{Hom}_{\mathbf{sSet}}(Y, \mathcal{C}^{p/}) \simeq \operatorname{Hom}_{\mathbf{sSet}_{/X}}(X \diamond Y, \mathcal{C})$, where $\diamond$ is a new operation such that $X \diamond Y \to X \star Y$ is a categorical equivalence. Now, we can form an equivalence $\mathcal{C}_{p/} \to \mathcal{C}^{p/}$, which sends initial object from the former to the latter. Therefore we can replace $\mathcal{C}_{p/}$ by $\mathcal{C}^{p/}$ in the definition of colimit.

- (Parametrized Colimit). The idea is that $\operatorname{colim} p$ should depend functorality on p. We define a notion $X \diamond_S Y$ for X, Y over S, that is compatible with base change: $(X_T \diamond_T Y_T) \simeq (X \diamond_S Y)_T$.
  - Let $K \in \mathbf{sSet}_{/S}$. The functor sending $X \in \mathbf{sSet}_{/S}$ to $X \diamond_S S \in (\mathbf{sSet}_{/S})_{K/}$ has a right adjoint, where a map $p_S : K \to Y$ over S is sent to $Y^{p_S/}$.
  - Let $q : Y \to S$ a Cartesian fibration and $p_S : K \to Y$ over S. Suppose that $p_s : K_s \to Y_s$ has a colimit in Y_s for each $s \in S$, and that $q \circ p_S$ is a coCartesian fibration, then there exists p'_S such that

$$\begin{array}{ccc} K & \xrightarrow{p_S} & Y \\ \downarrow & \overset{p'_S}{\nearrow} & \downarrow{\scriptstyle q} \\ K \diamond_S S & \longrightarrow & S \end{array}$$

    commutes, and that $p'_s : K_s \diamond \{s\} \to Y_s$ is a colimit of p_s. The collection of such $p'_s \in \operatorname{Fun}_S(K \diamond_S S, Y)$ is a contractible Kan complex.

- (Decomposition of Diagrams). We want to analyze $\operatorname{colim} p$ after restricting to some family of simplicial subsets K_I of K. We can also work in more general settings where each K_I just maps to K.
  - Now, fix $F : \mathcal{J} \to \mathbf{sSet}_{/K}$. We construct a simplicial set K_F such that there is a map $K_F \to \Delta^1$ where the fiber over 0 is K and over 1 is $N(\mathcal{J})$.
  - Suppose that for each I the map $p_I : K_I \to \mathcal{C}$ has a colimit q_I. There exists $\pi'_I : K_I^{\triangleright} \to K_F$ that commutes with π_I which sends the cone point to the vertex in $N(\mathcal{J})$ corresponds to I, then there exists $q : K_F \to \mathcal{C}$ such that $q \circ \pi'_I = q_I$ and $q_{|K} = p$. Moreover, $\mathcal{C}_{q/} \to \mathcal{C}_{p/}$ is a trivial fibration.
  - Now suppose $N(\mathcal{J}) \subset K_F$ is right anodyne, then $\mathcal{C}_{q/} \to \mathcal{C}_{p/}$ and $\mathcal{C}_{q/} \to \mathcal{C}_{q|_{N(\mathcal{J})}}$ are trivial fibrations, so in particular colimits of p can be identified with colimits of $q|_{N(\mathcal{J})}$.
- Suppose $\mathcal{C}$ and $\mathcal{J}$ are fibrant simplicial categories (the mapping spaces are Kan complexes). $C \in \mathcal{C}$ being a homotopy colimit of $F : \mathcal{J} \to \mathcal{C}$ (together with a collection of compatible maps $FI \to C$) in $\mathcal{C}$ iff after applying N and extending NF by maps $FI \to C$ to a map $N(\mathcal{J})^{\triangleright} \to N(\mathcal{C})$, we get a colimit diagram in $N(\mathcal{C})$.
- (Classical Kan Extension). Let $i : \mathcal{J} \to \mathcal{J}'$ be a functor between 1-categories. We have a natural functor $i^* : \mathcal{C}^{\mathcal{J}'} \to \mathcal{C}^{\mathcal{J}}$. The left adjoint to this functor is called the left Kan extension along i, and it exists if $\mathcal{C}$ has sufficiently many colimits.
- (Relative Colimit). Let $f : \mathcal{C} \to \mathcal{D}$ be an inner fibration and $\bar{p} : K^{\triangleright} \to \mathcal{C}$ a diagram. Let $p = \bar{p}_{|K}$. $\bar{p}$ is a f-colimit of p if $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$ is a trivial fibration (or equivalently, a categorical equivalence). In particular, when $\mathcal{D} = *$, this agrees with colimit, and when an edge $e : \Delta^1 \to \mathcal{C}$ is a f-colimit iff it is f-coCartesian.
  - Similar to colimit, relative colimits can be computed after precomposition with a cofinal map.
  - Let $q : X \to S$ locally coCartesian fibration. $\bar{p} : K^{\triangleright} \to X_s$ is a q-colimit (after composition with $X_s \to X$) iff for every edge $e : s \to s'$ in S, $e_! \circ \bar{p}$ is a colimit in $X_{s'}$. In particular, relative colimit can be detected fiberwise.
- (Kan Extensions along Inclusions). Let $\mathcal{C}^0 \subset \mathcal{C}$ be a full subcategory, and for a diagram $p : K \to \mathcal{C}$, we denote $\mathcal{C}^0_{/p} := \mathcal{C}_{/p} \times_{\mathcal{C}} \mathcal{C}^0$. In the following diagram where p an inner fibration,

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \downarrow & \overset{F}{\nearrow} & \downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

  we say F is a p-left Kan extension of F_0 if for every $C \in \mathcal{C}$ the extension

$$\begin{array}{ccc} \mathcal{C}^0_{/C} & \xrightarrow{F_C} & \mathcal{D} \\ \downarrow & \overset{\bar{F}_C}{\nearrow} & \downarrow p \\ (\mathcal{C}^0_{/C})^{\triangleright} & \longrightarrow & \mathcal{D}' \end{array}$$

  gives $F(C)$ as a p-colimit of F_C. In particular, if $C \in \mathcal{C}^0$ this is automatic. In the case $\mathcal{C} \to \mathcal{C}^{\triangleright}$, the map $\bar{F}$ is a left p-Kan extension iff it is a p-colimit of F.
  - A p-left Kan extension exists iff for every $C \in \mathcal{C}$, $\mathcal{C}^0_{/C} \to \mathcal{C}^0 \to \mathcal{D}$ admits a p-colimit.
  - In particular, if every fiber of p admits small colimits and $\mathcal{D}_E \to \mathcal{D}_{E'}$ preserves small colimits, then p-left Kan extension exists.

– Now suppose p is a categorical fibration, and every $F_0 \in \mathrm{Map}_{\mathcal{D}'}(\mathcal{C}', \mathcal{D})$ admits a p-left Kan extension, then the left Kan extension functor $i_!$ is left adjoint to the precomposition i^*.

- (Kan Extensions along General Functors). We focus on the absolute case. Note that in general, one do not expect the unit $f \to \delta^*\delta_! f$ to be an equivalence. We take this unit as part of the data. Let $\delta : K \to K'$ and $f : K \to \mathcal{D}$ a diagram. A left Kan extension of f along δ is a functor $f' : K' \to \mathcal{D}$ and $f \to f' \circ \delta$ in $\mathrm{Fun}(K, \mathcal{D})$.
  - Under this definition, $\delta_!$ is left adjoint to δ^*. Note that $\delta_!\delta^* F \to F$ does not need to be an equivalence as well. However, if δ is a fully faithful inclusion, then $f \to \delta^*\delta_! f$ is an equivalence.
- (Homotopy Coproduct). Let $\{X_\alpha\}$ be objects in a fibrant simplicial category $\mathcal{C}$. We can view the coproduct over X_α as the colimit for a functor from a discrete category A to $\mathcal{C}$. X is a homotopy coproduct if $\mathrm{Map}(X, Y) \simeq \prod \mathrm{Map}(X_\alpha, Y)$ as Kan complexes. We can also pass to the homotopy category $h\mathcal{C}$ and ask for an equivalence in $\mathcal{H}$.
  - In particular, suppose $p_\alpha : K_\alpha \to \mathcal{C}$ diagrams in an ∞-category $\mathcal{C}$ with colimits X_α. Let $K = \coprod K_\alpha$, then the colimit of $p = \coprod p_\alpha$ is $\coprod X_\alpha$ (and the former exists iff the latter does).
- (Pushouts). A square in an ∞-category $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$. Note that $(\Lambda^2_0))^{\triangleright} \simeq \Delta^1 \times \Delta^1 \simeq (\Lambda^2_2)^{\triangleleft}$. The square is a pushout or coCartesian (a pullback or Cartesian) if the corresponding diagram is a colimit (limit). Let $\mathcal{C}$ be a fibrant simplicial category, then a diagram is a homotopy pushout iff the square after applying Map is a pullback in **Kan**.
  - As in the classical case, given

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

    where the left square is a pushout, then the right square is a pushout iff the outer square is a pushout.
  - We can "decompose" colimits into a pushout of less complicated colimits. More specifically, let $p : K' = K \sqcup_L L' \to \mathcal{C}$, where $L \to L'$ is a monomorphism and colimits exist after restriction to L, L', and K (call them Z, X, and Y), then $\mathrm{colim}\, p = X \sqcup_Z Y$.
  - As in the classical case, an ∞-category admits all finite colimits iff it admits all pushouts and has an initial object. An ∞-functor preserves all finite colimits iff it preserves pushouts and an initial object.
  - Similarly, an ∞-category admits κ-small colimits if it admits κ-small coproducts and pushouts. An ∞-functor preserves all κ-small colimits iff it preserves pushouts and κ-small coproducts. (We can replace coproducts by coequalizers in the first sentence.)
- An ordinary category can be seen as tensored over **Set** if it admits coproducts. In particular, $S \otimes X := \coprod_S X$, and this satisfies $\mathbf{Set}(S \otimes X, Y) = \mathbf{Set}(S, \mathbf{Set}(X, Y))$ Analogously, an ∞-category is tensored over **Space** if it admits small colimits.
- Let $\mathcal{D}$ closed monoidal and $\mathcal{C}$ enriched over $\mathcal{D}$. A $\mathcal{D}$-enriched functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is representable if there exists $C \in \mathcal{C}$ and $\eta : 1_{\mathcal{D}} \to GC$ such that $\mathrm{Map}(X, C) \simeq \mathrm{Map}(X, C) \otimes 1_{\mathcal{D}} \to \mathrm{Map}(X, C) \otimes GC \to GX$ is an isomorphism in $\mathcal{D}$.
  - Now, let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Space}$ an ∞-functor. F is representable if $hF : h\mathcal{C} \to \mathcal{H}$ is representable. Here, $\eta : 1 \to FC$ can be regarded as $\eta \in \pi_0(FC)$.

- In particular, let $f : \tilde{\mathcal{C}} \to \mathcal{C}$ be a right fibration, then it is classified by some $F : \mathcal{C}^{\text{op}} \to \textbf{Space}$. Now, F is representable by (C, η) iff $\tilde{C}$ is final, where $C = f\tilde{C}$ and $\eta \in \pi_0 FC \simeq \pi_0(\tilde{\mathcal{C}} \times_{\mathcal{C}} C)$ contains $\tilde{C}$. We say f is representable if $\tilde{\mathcal{C}}$ has a final object.
- Let $p : K \to \mathcal{C}$ be a diagram with constant value X. We denote $X \otimes K$ to be the colimit of p if such colimit exists. This is well-defined up to unique weak equivalence and depends on the weak homotopy type of K.

- (Retracts and Idempotents). Let $\mathcal{C}$ be a category. If Y is a retract of X, i.e., there exists $\text{id} = i \circ r : Y \to X \to Y$, then we have in particular $f := r \circ i : X \to X$ idempotent, and Y can be viewed as a equalizer $(\text{id}_X, r \circ i)$. We have an injective map from the isomorphism classes of retracts of X to the set of idempotent maps $X \to X$. This is in particular a bijection (and we say $\mathcal{C}$ is idempotent complete) if $\mathcal{C}$ admits equalizers.
  - Now, in ∞-categorical setting, we say Y is a retract of X if it is the case in $h\mathcal{C}$, or if there exists a 2-simplex between $i \circ r$ and id. The analogous map still exists, but we cannot recover Y as a finite limit involving X.

- We can manually define simplicial sets Ret, Idem, Idem^+, where the latter two are ∞-categories, with natural maps $\text{Ret} \to \text{Idem}^+$ and $\text{Idem} \to \text{Idem}^+$. We call an idempotent (weak retraction diagram, strong retraction diagram) in $\mathcal{C}$ to be a functor from Idem (Ret, Idem^+) to $\mathcal{C}$.
  - $\text{Fun}(\text{Idem}^+, \mathcal{C}) \to \text{Fun}(\text{Ret}, \mathcal{C})$ is a trivial fibration, so every weak retraction diagram can be extended to a strong retraction diagram.
  - An idempotent is effective if it can be extended to a strong retraction diagram, iff the functor admits a limit/colimit. $\mathcal{C}$ is said to be idempotent complete if every idempotent is effective.
  - A strong retraction diagram can be seen as the Kan extension of the restriction to an idempotent. If $\mathcal{C}$ is idempotent complete, then $\text{Fun}(\text{Idem}^+, \mathcal{C}) \to \text{Fun}(\text{Idem}, \mathcal{C})$ is a trivial fibration. In particular, every idempotent can be extended to a strong retraction diagram iff this map is a trivial fibration.

6.5 Presentable and Accessible ∞-Categories

- (Yoneda Lemma). The classical Yoneda lemma says that $\mathcal{C} \to \text{Hom}(\mathcal{C}^{\text{op}}, \textbf{Set})$ is fully faithful. In ∞-categorical setting, we replace the target from presheaves of sets to presheaves of spaces. For a simplicial set S, we denote $\text{Fun}(S^{\text{op}}, \textbf{Space})$ to be $\mathcal{P}(S)$, referred to as the ∞-category of presheaves on S. We in fact have a map $S \to \mathcal{P}(S)$ which is also fully faithful. In addition, the map preserves all small limits.
  - There are other models for $\mathcal{P}(S)$ that are canonically equivalent to each other. For example, $N(s\mathcal{P}'(S))$ where $s\mathcal{P}'(S)$ is the full subcategory of $\textbf{sSet}_{/S}$ spanned by right fibrations over S, or $N(s\mathcal{P}''(S))$ where $s\mathcal{P}''(S)$ is the simplicial subcategory of simplicial presheaves over $\mathbb{C}[S]$ spanned by fibrant-cofibrant objects.

- $\mathcal{P}(S)$ admits all small limits and colimits.

- (Idempotent Completion). We say f exhibits $\mathcal{D}$ as an idempotent completion for $\mathcal{C}$ if $f : \mathcal{C} \to \mathcal{D}$ is fully faithful, $\mathcal{D}$ is idempotent complete, and that every object in $\mathcal{D}$ is a retract of an object in $f\mathcal{C}$.
  - We can define an idempotent completion by considering the subcategory of $\mathcal{P}(\mathcal{C})$ spanned by retracts. In particular, it is unique in the sense that if $\mathcal{E}$ is idempotent complete, then $\text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$ is an equivalence of ∞-categories.

- We can informally view $\mathcal{P}(S)$ as the limit of the diagram $S^{\text{op}} \to \infty\textbf{Cat}$ where everything is sent to $\textbf{Space}$. In addition we have $\text{Fun}(\mathcal{C}, \mathcal{P}(S)) \simeq \text{Fun}(\mathcal{C} \times S^{\text{op}}, \textbf{Space})$.

- Let $\mathrm{Fun}^L(\mathcal{D}, \mathcal{C})$ be the functors that are left adjoints. When $\mathcal{D} = \mathcal{P}(S)$, such functors are exactly those which preserve colimits. (Suppose $\mathcal{C}$ admits small colimits) Every functor $S \to \mathcal{C}$ extends to a colimit-preserving functor $\mathcal{P}(S) \to \mathcal{C}$. In other words, $\mathrm{Fun}^L(\mathcal{P}(S), \mathcal{C}) \to \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C})$ is an equivalence.
  - Colimits-preserving functors $f : \mathcal{P}(S) \to \mathcal{C}$ are exactly those that are left Kan extensions of their restrictions to the essential image of the Yoneda embedding. In particular, $F \simeq \mathrm{colim}_{(j(s)\to F)\in \mathcal{P}(S)^{\mathrm{rep}}_{/F}} j(s)$ where j is the Yoneda embedding.
  - Say $\mathcal{C}' \subset \mathcal{C}$ is stable under colimits if for any small diagram $K \to \mathcal{C}'$ which has a colimit $K^{\triangleright} \to \mathcal{C}$, the colimit factors through $\mathcal{C}'$. A collection of objects A generates $\mathcal{C}$ under colimits if any full subcategory $\mathcal{C}' \subset \mathcal{C}$ that is stable under colimits and contains A is equal to $\mathcal{C}$. In particular, $\mathcal{P}(S)$ is generated by S under small colimits via the Yoneda embedding.
- Let $f : S \to \mathcal{C}$ be a diagram, we want to know when $\mathcal{C} \simeq \mathcal{P}(S)$. If $\mathcal{C}$ admits small colimits, we can extend this map to $\mathcal{P}(S) \to \mathcal{C}$, and the extension is an equivalence iff f is fully faithful, generates $\mathcal{C}$ under colimits, and that the essential image of f consists of complete compact objects of $\mathcal{C}$.
- Let $\mathcal{M} \to \Delta^1$ be a Cartesian fibration, this can be viewed as a functor $(\Delta^1)^{\mathrm{op}} \to \infty\mathbf{Cat}$. We in particular get a functor $\mathcal{D} \to \mathcal{C}$ between ∞-categories. There is a bijection between functors $\mathcal{D} \to \mathcal{C}$ and equivalence classes of Cartesian fibrations $\mathcal{M} \to \Delta^1$ with equivalences $\mathcal{C} \to \mathcal{M}_{\{0\}}$ and $\mathcal{D} \to \mathcal{M}_{\{1\}}$.
- (Adjunction). An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a map $p : \mathcal{M} \to \Delta^1$ that is both a Cartesian and a coCartesian fibration, with equivalences $\mathcal{C} \to \mathcal{M}_{\{0\}}$ and $\mathcal{D} \to \mathcal{M}_{\{1\}}$. Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ to be functors associated to p, we say f is left adjoint to g and vice versa.
  - According to the above, if f admits a right adjoint, then it is unique up to equivalences.
  - Adjunction is preserved by compositions. Adjoints can be determined fiberwise in the sense that $X \to S$ is a coCartesian fibration iff all the maps $X_{s'} \to X_s$ have left adjoints.
  - Adjunction can also be characterized by the existence of a unit in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.
  - If f is a left adjoint then so is hf. Conversely, if hf admits a right adjoint then so does f.
- Left adjoints preserve colimits, and right adjoints preserve limits.
- Let $\mathcal{C}$, $\mathcal{D}$ fibrant simplicial categories with $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ adjoints. Then the induced functors NF and NG are adjoints.
  - In particular, let $\mathbf{A}$ and $\mathbf{A}'$ be simplicial model categories with F, G a pair of a simplicial Quillen adjunction. Let $\mathcal{M}$ be the correspondence between (F, G), and $\mathcal{M}^\circ$ be the full subcategory of fibrant-cofibrant objects. Then $N\mathcal{M}^\circ$ gives an adjunction between $N\mathbf{A}^\circ$ and $N\mathbf{A}'^\circ$. We say the induced adjunction between $N\mathbf{A}^\circ$ and $N\mathbf{A}'^\circ$ to be the left and right derived functors of the Quillen adjunction.
- Assume $\mathcal{C}$ admits pullbacks and $C \in \mathcal{C}$, let (F, G) be an adjunction between $\mathcal{C}$ and $\mathcal{D}$. Then the induced functor $\mathcal{C}^{C/} \to \mathcal{D}^{FC/}$ admits a right adjoint $\mathcal{D}^{FC/} \to \mathcal{C}^{GFC/} \to \mathcal{C}^{C/}$ where the second map is induced by the unit $C \to GFC$.
- Note that we have a colimit-preserving map $\mathcal{P}(f) : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$ given a map $f : \mathcal{C} \to \mathcal{C}'$, which corresponds to the composition of f with the Yoneda embedding: $\mathcal{C}' \to \mathcal{P}(\mathcal{C}')$. $\mathcal{P}(f)$ is the left adjoint to the natural map $\mathcal{P}(\mathcal{C}') \to \mathcal{P}(\mathcal{C})$.

- Suppose we want to localize **Ab** by maps with kernel and cokernel p-torsion. We can take the full subcategory of uniquely p-divisible groups, and this gives a localization of **Ab** with the size controlled. The localization is given by $A \to A \otimes \mathbb{Z}[1/p]$, and it is left adjoint to the inclusion. We can do the similar for **Sp** via the Bousfield localization by tensoring with $\mathbb{S}[1/p]$. We want to do this for ∞-categories in general.

- (Localization). A functor $f : \mathcal{C} \to \mathcal{D}$ is a localization if it admits a fully faithful right adjoint g. We identify $\mathcal{D}$ with its image in $\mathcal{C}$ via g, and let $L = g \circ f : \mathcal{C} \to \mathcal{C}$.
  - L is a localization iff L is left adjoint to the inclusion $L\mathcal{C} \to \mathcal{C}$, iff there exists a natural transformation $\mathcal{C} \times \Delta^1 \to \mathcal{C}$ from id to L such that $LC \to LLC$ is an equivalence for each $C \in \mathcal{C}$.
  - Localization satisfies the universal property in the sense that $\mathrm{Fun}(L\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, and the essential image contains those functors $\mathcal{C} \to \mathcal{D}$ where any map f in $\mathcal{C}$ such that Lf is an equivalence is sent to an equivalence in $\mathcal{D}$.

- (Orthogonal Maps and Factorization System). A map of set can be (uniquely) factorized as the composition of a surjective map followed by an injective map. We want to do the same for ∞-categories. In particular, say $f \perp g$ if for every commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

  $\mathrm{Map}_{\mathcal{C}_{A//Y}}(B, X)$ is contractible. Or in other words, the space of lifts $B \to X$ is contractible.
  - Now, (S_L, S_R) is called a factorization system where S_L and S_R are collections of morphisms in $\mathcal{C}$ if every map in S_L is left orthogonal to every map in S_R, they are closed under retracts, and any map in $\mathcal{C}$ can be factored to a composition of a map in S_L and a map in S_R up to a 2-homotopy. In fact, $S_L = {}^{\perp} S_R$ and $S_R = S_L^{\perp}$.
  - In fact, such factorizations are canonical, in the sense that $\mathrm{Fun}'(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ is a trivial Kan fibration, where the former represents the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ where the upper two edges are sent to maps in S_L and S_R.

- Filtered topological or fibrant simplicial categories are defined analogous to how filtered categories are defined.

- An ∞-category $\mathcal{C}$ is κ-filtered if for every κ-small simplicial set K and a map $K \to \mathcal{C}$, there exists an extension $K^{\triangleright} \to \mathcal{C}$. An ω-filtered ∞-category is said to be filtered. In particular, a topological category is filtered iff its nerve is.
  - In other words, every κ-small diagram in a κ-small category admits an upper bound.

- If $\mathcal{C}$ is κ-filtered then there exists a κ-filtered poset A with $NA \to \mathcal{C}$ cofinal. $\mathcal{C}$ is κ-filtered iff $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is cofinal for every κ-small simplicial set K.

- (Right Exact Functors). For Abelian categories admitting finite colimits, a functor is right exact iff it preserves finite colimits. For a functor between ∞-categories, F is κ-right exact if for any right fibration $\mathcal{B}' \to \mathcal{B}$ where $\mathcal{B}'$ is κ-filtered, $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}'$ is κ-filtered. In particular, if $\mathcal{A}$ admits κ-small colimits, then a functor is κ-right exact iff it preserves κ-small colimits.

- An ∞-category is κ-closed if every diagram $p : K \to \mathcal{C}$ where K is κ-small admits a colimit.
  - This gives a way for defining the "colimit functor". More generally, let $\mathcal{C}$ be an category and K a simplicial set, such that every map $K \to \mathcal{C}$ has a colimit $K^{\triangleright} \to \mathcal{C}$, then we can define $\mathcal{D} \subset \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ the subcategory spanned by colimit diagrams. Now, we can define $\mathrm{colim}_K : \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{D} \subset \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathcal{C}$, where the first map comes from the section of the trivial fibration $\mathcal{D} \to \mathrm{Fun}(K, \mathcal{C})$ and the last map comes from evaluating at the cone point. This gives a colimit in $\mathcal{C}$ for each diagram.

- $\mathcal{J}$ is κ-filtered iff the colimit functor $\operatorname{colim}_{\mathcal{J}} : \operatorname{Fun}(\mathcal{J}, \mathbf{Space}) \to \mathbf{Space}$ preserves κ-small colimits.

- (Compact Objects). For ordinary categories, an object C is compact if $\operatorname{Hom}(C, -)$ commutes with filtered colimits. Analogously, in an ∞-category $\mathcal{C}$, an object C is κ-compact if the corepresentable functor $j_C : \mathcal{C} \to \widehat{\mathbf{Space}}$ is κ-continuous (i.e., preserves κ-filtered colimits) where the latter denotes the category of large spaces. A left fibration $\tilde{\mathcal{C}} \to \mathcal{C}$ is κ-compact if it is classified by a κ-continuous $\mathcal{C} \to \widehat{\mathbf{Space}}$.

- (Ind-Object). We call an Ind-object of a category $\mathcal{C}$ to be a diagram $f : \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a filtered category. Or we can identify f with $\operatorname{colim} X_i$. This gives a category $\operatorname{Ind}(\mathcal{C})$ which contains $\mathcal{C}$ as a full subcategory. Moreover, any map $\mathcal{C} \to \mathcal{D}$ that preserves filtered colimits can be extended to a map $\operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$ whose restriction to $\mathcal{C}$ is isomorphic to the map we begin with, and this extension is unique up to an unique isomorphism. We want to do the same for ∞-categories.
  - Let $\mathcal{C}$ be the category of finitely presented groups, then $\operatorname{Ind}(\mathcal{C})$ is the category of groups, in the sense that every group is a filtered colimit of finitely presented groups.
  - Note that $\mathcal{P}(\mathcal{C})$ is an ∞-category freely generated by $\mathcal{C}$ under colimits. We want an analogous construction with filtered colimits. A hope might be to find $\operatorname{Ind}(\mathcal{C})$ inside $\mathcal{P}(\mathcal{C})$: $\operatorname{Ind}_\kappa(\mathcal{C})$ is the full subcategory of $\mathcal{P}(\mathcal{C})$ where the functors are classified by right fibrations $\tilde{\mathcal{C}} \to \mathcal{C}$ where $\tilde{\mathcal{C}}$ is κ-filtered. It is stable under κ-filtered colimits.
  - Suppose $\mathcal{C}$ admits κ-small colimits, then $F \in \operatorname{Ind}_\kappa(\mathcal{C})$ iff F preserves κ-small limits. In addition, $j(C)$is κ-compact in $\operatorname{Ind}_\kappa(\mathcal{C})$.
  - In fact, suppose $\mathcal{D}$ admits κ-filtered colimits, there is an equivalence $\operatorname{Map}_\kappa(\operatorname{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ where the LHS is the ∞-category of all κ-continuous functors. Similar to what we did for $\mathcal{P}(S)$, we can express every $F \in \operatorname{Ind}_\kappa(\mathcal{C})$ as a κ-filtered colimit (which is essentially via the definition, because $\tilde{\mathcal{C}} \simeq \mathcal{C}^{\text{rep}}_{/F}$).

- This is actually an universal construction, in the sense that we can replace filtered by any collection of diagrams:
  - Let $\mathcal{K}$ be a collection of simplicial sets. An ∞-category admits $\mathcal{K}$-indexed colimits if it admits K-indexed colimits for each $K \in \mathcal{K}$. $\operatorname{Fun}_{\mathcal{K}}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors which preserve $\mathcal{K}$-indexed colimits.
  - Let $\mathcal{R}$ be a collection of diagrams $K_\alpha^{\triangleright} \to \mathcal{C}$. $\operatorname{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors that send diagrams in $\mathcal{R}$ to colimit diagrams in $\mathcal{D}$.
  - There exists an ∞-category $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ and $j : \mathcal{C} \to \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ such that:
    * $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ admits $\mathcal{K}$-indexed colimits.
    * If $\mathcal{D}$ admits $\mathcal{K}$-indexed colimits, then there is an equivalence $\operatorname{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}), \mathcal{D}) \to \operatorname{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$.
    * If every diagram in $\mathcal{R}$ is a colimit, then j is fully faithful.

- (Essentially Small). An ∞-category $\mathcal{C}$ is essentially κ-small if the collection of equivalence classes of objects in $\mathcal{C}$ is κ-small, and for every $f : C \to D$, $\pi_i(\operatorname{Hom}^R(C, D), f)$ is κ-small.
  - Let $\mathcal{C} \to \mathcal{D}$ be a functor where the latter is essentially κ-small, to conclude that $\mathcal{C}$ is essentially κ-small, it suffices to have $\mathcal{C}_D$ essentially κ-small for each $D \in \mathcal{D}$.
  - An Kan complex is essentially κ-small if $\pi_i(X, x)$ is κ-small for each i and x.
  - Being essentially κ-small is equivalent to being κ-compact in $\infty\mathbf{Cat}$ or $\mathbf{Space}$.

- (Locally Small). Relax the conditions a bit, we say an ∞-category is locally small if $\operatorname{Map}(X, Y)$ is essentially small for each pair of objects, or if the full subcategory spanned by any small collection of objects is essentially small.

- (Accessible ∞-Category). The motivation: for an ordinary category $\mathcal{C}$, a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ leads to a fibered category $\tilde{\mathcal{C}} \to \mathcal{C}$. This fibered category is equivalent to $\mathcal{C}_{/C}$, where C is the colimit of the map $\tilde{\mathcal{C}} \to \mathcal{C}$ if $\mathcal{C}$ admits colimits. Here set-theoretical issues arise: if we just assume $\mathcal{C}$ admits small colimits then $\tilde{\mathcal{C}}$ is not necessarily small, while if we assume $\mathcal{C}$ small it is not reasonable to expect $\mathcal{C}$ admit small colimits. An accessible category is not small but controlled by some small subcategory $\mathcal{C}^0$, in the sense that $\mathcal{C}' \times_{\mathcal{C}} \mathcal{C}^0$ is small so the colimit is expected to exist. For the ∞-categorical setting, we define an κ-accessible ∞-category to be equivalent to $\mathrm{Ind}_\kappa(\mathcal{C}^0)$ for some small ∞-category $\mathcal{C}^0$. We say $\mathcal{C}$ is accessible if it is κ-accessible for some κ.
  - $\mathcal{C}$ is κ-accessible if it is locally small and admits κ-filtered colimits, and the full subcategory $\mathcal{C}^\kappa$ of κ-compact objects is essentially small and generates $\mathcal{C}$ under small, κ-filtered colimits.
  - **Space** and $\mathcal{P}(\mathcal{C})$ are accessible for $\mathcal{C}$ small.
- (Accessible Functor). Let $\mathcal{C}$ be accessible. A functor $F : \mathcal{C} \to \mathcal{C}'$ is accessible if it is κ-continuous for some regular cardinal κ.
- If $\mathcal{C}$ is κ-accessible, then it is κ'-accessible for any $\kappa' \gg \kappa$ but generally not for $\kappa' > \kappa$.
- The category of κ-accessible ∞-categories is equivalent to the category spanned by idempotent complete ∞-categories, and idempotent completion denotes the former as a localization of the category of ∞-categories. In fact, we know that a small ∞-category is an accessible iff it is idempotent complete.
- Let $\mathcal{C}$ accessible and K small simplicial set, then $\mathrm{Fun}(K, \mathcal{C})$ is also accessible. If $p : K \to \mathcal{C}$ is a diagram, then $\mathcal{C}_{p/}$ is accessible (and same for $\mathcal{C}_{/p}$). The class of accessible categories is stable under fiber products.
- (Presentable ∞-Category). An ∞-category is presentable if it is accessible and admits small colimits.
  - An ∞-category is presentable iff it is an accessible localization (i.e., an localization such that L is accessible) of $\mathcal{P}(\mathcal{D})$ for some $\mathcal{D}$ small. In fact an localization L of an accessible category $\mathcal{C}$ is accessible iff the essential image $L\mathcal{C}$ is accessible.
  - If $\mathcal{C}$ is κ-accessible, then $\mathcal{C}$ is presentable iff the full subcategory $\mathcal{C}^\kappa$ admits κ-small colimits. Roughly speaking this is because small colimits can be written as κ-filtered colimits and κ-small colimits of κ-compact objects. Similarly, assume $\mathcal{C}$ and $\mathcal{D}$ accessible, and the former κ-accessible, $f : \mathcal{C} \to \mathcal{D}$ preserves small colimits iff it is κ-continuous and $f|_{\mathcal{C}^\kappa}$ preserves κ-filtered colimits.
- $F \in \mathcal{P}(\mathcal{C})$ is representable if it is in the essential image of the Yoneda embedding. Now if F is representable, it preserves small limits. If $\mathcal{C}$ is presentable, the converse is true. So a presentable ∞-category admits all small limits.
  - F is corepresentable iff F is accessible and preserves small limits.
  - (Adjoint Functor Theorem). A functor between presentable ∞-categories has a right (left) adjoint iff it preserves small colimits (limits and is in addition accessible).
- We define $\mathbf{Pr^L}$, $\mathbf{Pr^R} \subset \infty\mathbf{Cat}$ to be the subcategories of presentable ∞-categories, where functors in the former are those that preserve small colimits, and the functors in the latter are those that are accessible and preserve small limits.
- A presentable fibration is a morphism that is both Cartesian and coCartesian with the fibers presentable ∞-categories. The equivalence classes of presentable fibrations $X \to S$ can be identified with $[S, \mathbf{Pr^L}]$ or with $[S^{\mathrm{op}}, \mathbf{Pr^R}]$, and in particular $\mathbf{Pr^L}$ and $\mathbf{Pr^R}^{\mathrm{op}}$ are equivalent in the homotopy category of $\infty\mathbf{Cat}$.

- Presentable categories are stable under many basic constructions, including:
  - products,
  - $\mathrm{Fun}(K, -)$'s for K small simplicial set,
  - $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ (which can be viewed as the internal hom-set in $\mathbf{Pr^L}$),
  - $\mathcal{C}_{/p}$ for p small diagram,
  - fiber products with morphisms in $\mathbf{Pr^L}$.
- $\mathbf{Pr^L}$ and $\mathbf{Pr^R}$ admit small limits and the inclusion functors preserve all small limits. Essentially, limits and colimits (first dualize to $\mathbf{Pr^R}$ and then compute as limits) can be computed in $\infty\mathbf{Cat}$.
- A localization is determined up to equivalence by the collection of morphisms S that are mapped to equivalences. Such a collection arises from a accessible localization iff it is strongly saturated and of small generation. Moreover, given any collection S, we can find the smallest strongly saturated collection that contains S. This allows us to define $S^{-1}\mathcal{C}$ for any collection of morphisms S.
  - For any S, the full subcategory $S^{-1}\mathcal{C}$ of S-local objects is a localization of $\mathcal{C}$ and every localization arises that way. $S^{-1}\mathcal{C}$ and $T^{-1}\mathcal{C}$ coincide iff they generate the same strongly saturated collection of morphisms. (Suppose $\mathcal{C}$ accessible) The localization is accessible iff $S^{-1}\mathcal{C}$ is, or iff there exists a small subset of S that generates the same strongly saturated class of maps.
  - As expected, let $\mathcal{C}$ presentable, then $\mathrm{Fun}^L(S^{-1}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, with the essential image being those that send S to equivalences.
  - Note that $L\mathcal{C}$ is accessible iff it is presentable.
- A similar phenomena holds for factorization systems: Let $\mathcal{C}$ presentable, S a saturated collection of maps that is also of small generation, then $(S, S^{\perp})$ gives a factorization system. Note that the converse need not hold.
- (Truncated Objects). Recall that in classical topology, we say a space X is k-truncated if all its homotopy groups of dimension higher than k vanish. A nice space X admits an essentially unique Postnikov tower: $X \to ... \to \tau_{\leq n}X \to ... \to \tau_{\leq -1}X$, where $\tau_{\leq k}X$ is k-truncated and is universal. $\tau_{\leq 0} = \pi_1 X$ and $\tau_{\leq 1}X = B\Pi X$. X can be recovered by taking homotopy limit of the tower. We can do the same for an object in an ∞-category.
  - A Kan complex X is k-truncated if $\mathrm{Map}(S, X)$ is for every small simplicial set S. $C \in \mathcal{C}$ is k-truncated if for every $D \in \mathcal{C}$, $\mathrm{Map}(D, C)$ is.
  - A morphism is k-truncated if $\mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ is for every E.
- Let $\mathcal{C}$ presentable, $\tau_{\leq k}\mathcal{C} \subset \mathcal{C}$ admits an accessible left adjoint, which is denoted as $\tau_{\leq k}$.
- **Space** is generated by compact projective objects under colimits. Having enough compact projective objects allows us to define projective resolution, which gives rise to non-abelian homological (or homotopical) algebra.
- An ∞-category is κ-compactly generated if it is presentable and κ-accessible; in particular, iff there exists some small ∞-category $\mathcal{D}$ that admits κ-small colimits, and that $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{D})$.
  - A localization L is κ-continuous iff $L\mathcal{C}$ is stable under κ-filtered colimits. If such, then $L\mathcal{C}$ is κ-compactly generated, and L carries κ-compact objects to κ-compact objects.

7 Characteristic Classes

7.1 Vector Bundles

- A n-dim. v.b. is trivial iff there are n linearly independent sections. For example, S^1 and S^3 are parallelizable.
  - We denote the tautological bundle over $\mathbb{P}^n$ to be γ_n^1.
    - $E(\gamma_n^1)$ is exactly the Möbius band so $\mathbb{P}^1$ is not parallelizable.
- A v.b. is Euclidean if there exists continuous $\mu : E \to \mathbb{R}$ such that the restriction to each fiber is positive definite and quadratic.
  - An Euclidean metric on TM would be the same as a Riemannian metric.
- (Whitney Sum): Let ξ_1, ξ_2 be two v.b.'s over B. We define $\xi_1 \oplus \xi_2$ via $\Delta^*(\xi_1, \xi_2)$ where Δ is the diagonal map. Note that each fiber is exactly the direct sum of the two summands' fibers.
  - $\xi_x \simeq (\xi_1)_x \oplus (\xi_2)_x$ implies $\xi \simeq \xi_1 \oplus \xi_2$.
  - Let $\xi \subset \eta$ be two v.b.'s over X with Euclidean metric on η. We can define $\xi^\perp$ fiberwise, and $\eta \simeq \xi \oplus \xi^\perp$.
  - In particular, let $M \subset N$ Riemannian, then $TM \oplus \nu \simeq f^*TN$.
  - $\bigwedge^k V$ is defined fiberwise, and is of rank C_n^k.
  - In fact, given a functor $\mathbf{Vect}_\mathbb{R} \times ... \times \mathbf{Vect}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$, we can always define a functor from k vector bundles to a vector bundle fiberwise, and the topology on the total space is canonical.

7.2 Stiefel-Whitney Class

- We can define the Stiefel-Whitney classes $(w_i(\xi)) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ for a vector bundle $\xi \to B$ such that
  - w_i is natural. $w_k(\xi) = 0$ if $k > \text{Rank}(\xi)$.
  - $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}/2\mathbb{Z})$.
  - $w_k(\xi \oplus \eta) = \sum w_i(\xi) \cup w_j(\eta)$.
  - $w_1(\gamma_1^1) \neq 0$.

  In particular, these indicate:
  - $w_i(\epsilon) = 0$ if ϵ trivial and $i > 0$.
  - If ξ has k linearly independent sections, then $w_n(\xi) = ... = w_{n-k+1}(\xi) = 0$.
  - Let $H^\Pi(B; \mathbb{Z}/2\mathbb{Z})$ be the ring with all formal infinite series $a = a_1 + ...$, then $w(\xi)$ has an inverse, denoted $\bar{w}(\xi)$ and $w(\xi) = \bar{w}(\eta)$ if $\xi \oplus \eta$ is trivial. In particular, $w(\nu) = \bar{w}(TM)$ for $M \to N$ embedding.
  - $w(S^n) = \bar{w}(\nu_{S^n}) = 1$.
  - $H^i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ with generator $a_i = a_1^i$ for $0 \leq i \leq n$ and $w(\gamma_n^1) = 1 + a$. $w(\gamma^\perp) = 1 + a + ... + a^n$.
- $T\mathbb{RP}^n \simeq \text{Hom}(\gamma_n^1, \gamma^\perp)$. In particular, $T\mathbb{RP}^n \oplus \epsilon^1 = \text{Hom}(\gamma_n^1, \gamma^\perp) \oplus \text{Hom}(\gamma_n^1, \gamma_n^1) = (\gamma_n^1)^{n+1}$. Therefore, $w(\mathbb{RP}^n) = (1 + a)^{n+1}$.
  - It is easy to show that $w(T\mathbb{RP}^n) = 1$ iff $n + 1 = 2^k$; this shows that most projective spaces are not parallelizable.

– This also shows that the bound for the Whitney immersion theorem is optimal.

- Let $[M] \in H_n(M;\mathbb{Z}/2\mathbb{Z})$ be a fundamental class, we can define $H^n(M;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ via evaluating at $[M]$. The Stiefel-Whitney numbers are defined to be $\langle w_1(TM)^{r_1}...w_n(TM)^{r_n}, [M]\rangle$ for each $(r_1, ..., r_n)$ that makes the dimension right.
  - In fact one can show that M can be realized as the boundary of some smooth compact manifold iff all Whitney numbers vanish.
- $w_1(\xi) = 0$ iff ξ is orientable: we can classify ξ as a map $f \in [X, BO(n)]$, and to lift f to $[X, BSO(n)]$, it suffices to show that its image in $[X, B\mathbb{Z}/2\mathbb{Z}]$ is zero via the SES of groups $SO(n) \to O(n) \to \mathbb{Z}/2\mathbb{Z}$. Now, via some diagram chasing we can show that it reduces to $w_1(\xi) = 0$.
- $\mathrm{Spin}(n) \to SO(n)$ is the universal cover. We now have a fiber sequence $B\,\mathrm{Spin}(n) \to BSO(n) \to K(\mathbb{Z}/2\mathbb{Z}, 2)$ and to lift $f \in [X, BSO(n)]$ to $[X, B\,\mathrm{Spin}(n)]$, we need exactly the composition in $[X, K(\mathbb{Z}/2\mathbb{Z}, 2)]$ to be zero, or if $w_1(\xi) = w_2(\xi) = 0$, so this happens iff ξ has a spin structure.
- Let M^1 or $M^k \subset \mathbb{R}^{k+1}$, we can define natural maps $M^1 \to S^k$ or $M^k \to S^k$. Generalizing these gives $M^k \subset \mathbb{R}^{n+k}$ a map $M^k \to \mathrm{Gr}_k(\mathbb{R}^{n+k}) \simeq \mathrm{Gr}_n(\mathbb{R}^{n+k})$. This leads to a map of v.b.'s $TM \to \gamma^n(\mathbb{R}^{n+k})$. In fact, we can do this for every rank n ξ over M assuming compactness conditions. Taking colimits gives the infinite Grassmanian Gr_n, which classifies any rank n bundles up to homotopy.
- We now attempt to compute the $\mathbb{Z}/2\mathbb{Z}$ cohomology ring of Gr_n.
  - (Cell Structure). Let a Schubert symbol $\sigma = (\sigma_1, ..., \sigma_n)$ be a sequence of integers $1 \le \sigma_1 < ... < \sigma_n \le m$ and $e(\sigma) \subset \mathrm{Gr}_n(\mathbb{R}^m)$ be the collection of $V \in \mathrm{Gr}_n(\mathbb{R}^m)$ such that $\dim X \cap \mathbb{R}^{\sigma_i} = \dim X \cap \mathbb{R}^{\sigma_{i-1}} + 1 = i$. In other words, the sequence of integers $(\dim X \cap \mathbb{R}^j)_{1\le j\le m}$ jumps exactly at each σ_i.
    * Note that each $X \in e(\sigma)$ contains an unique orthonormal basis $(x_1, ..., x_n) \in \mathbb{H}^{\sigma_1} \times ... \times \mathbb{H}^{\sigma_n}$. It can then be shown that $e(\sigma)$ is an open cell of dimension $\sigma_1 - 1 + \sigma_2 - 2 + ... + \sigma_n - n$.
    * It can then be shown that the $\binom{m}{n}$ cells $e(\sigma)$ gives a cell structure on $\mathrm{Gr}_n(\mathbb{R}^m)$, and letting $m \to \infty$ gives a cell structure on $\mathrm{Gr}_n(\mathbb{R}^\infty)$.
    * We can also show that the number of r-cells in $\mathrm{Gr}_n(\mathbb{R}^m)$ is the number of partitions of r into at most n integers each no greater than $m - n$.
- We now claim that $H^*(\mathrm{Gr}_n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, ..., w_n]$ where each $w_i = w_i(\gamma^n)$.
  - To show that it contains the polynomial algebra, we need to show the w_i's are algebraically independent. It suffices to show that after pulling back to some rank n bundle, for example the n-fold product of γ_n^1 on $\mathbb{RP}^\infty$. It's easy to see that $w(\prod \mathbb{RP}^n) = (1 + a_1)...(1 + a_n)$ so that each w_k is the symmetric function of degree k, and they are algebraically independent.
  - Now we can show the two algebra agree by counting the dimensions: the number of r-cells is bounded, so is the rank of the r-th cohomology group according to cellular cohomology.
- (Thom Isomorphism). There exists $u \in H^n(E, E_0;\mathbb{Z}/2\mathbb{Z})$ such that u restricts to the unique non-zero class in each fiber (F, F_0). Cup product with u gives an isomorphism $H^i(E) \simeq H^{i+n}(E, E_0)$. Now the Thom isomorphism is the composition of isomorphisms $H^i(B) \to H^i(E) \to H^{i+n}(E, E_0)$.
- (Definition of the Stiefel-Whitney Class). We assume there are Steenrod operations $\mathrm{Sq}^i : H^n(X, Y) \to H^{n+i}(X, Y)$ that satisfies various axioms. We define $w_i(\xi) = \phi^{-1}\mathrm{Sq}^i(u)$ where ϕ is the Thom isomorphism.

7.3 Orientation and Euler Class

- For an oriented vector space V we can define a generator $\mu_V \in H_n(V, V_0; \mathbb{Z})$ and hence a generator $u_V \in H^n(V, V_0; \mathbb{Z})$. Therefore, for an oriented bundle, we can find a generator $u_F \in H^n(F, F_0; \mathbb{Z})$ for each fiber F, and locally the choice of these u_F's should glue to a cohomology class u.
  - In fact we can show that there uniquely exists globally $u \in H^n(E, E_0; \mathbb{Z})$ that restricts to the generator of each fiber. Cup product with u gives an isomorphism $H^k(E; \mathbb{Z}) \simeq H^{n+k}(E, E_0; \mathbb{Z})$.
  - The proof idea: we show that $H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \simeq H^0(B)$, and the class $1 \times e^n$ corresponding to $1 \in H^0(B)$ satisfies the condition. Now, we can glue two classes together via Mayer-Vietoris.
- (Euler Class). We define $e(\xi) \in H^n(B; \mathbb{Z})$ to be the class that corresponds to $u|_E \in H^n(E; \mathbb{Z})$ via the isomorphism $H^n(B; \mathbb{Z}) \simeq H^n(E; \mathbb{Z})$ (given by the deformation retract).
  - $e(\xi)$ is natural w.r.t. orientation-preserving bundle maps and changes sign if the orientation is reversed.
  - We can see that $\phi(e(\xi)) = u \cup u$ where ϕ is the Thom isomorphism. In particular, we know that $2e(\xi) = 0$ if $\mathrm{Rank}(\xi)$ is odd.
  - Since $u \cup u = \mathrm{Sq}^n(u)$, the image of $e(\xi)$ under the map $H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}/2\mathbb{Z})$ is $w_n(\xi)$.
  - $e(\xi \oplus \eta) = e(\xi) \cup e(\eta)$ and $e(\xi \times \eta) = e(\xi) \times e(\eta)$.
  - If ξ has a nowhere zero section, then $e(\xi) = 0$.
- A complex vector space has a canonical real orientation, because any complex basis $(a_j + ib_j)$ gives a real basis $(a_1, b_1, a_2, ..., a_n, b_n)$ and any complex linear automorphism respects this real orientation. In particular, any complex n-bundle can be viewed as a real $2n$-bundle and is automatically real orientable.
- Let $M^n \subset A^{n+k}$.
  - (Tubular Neighborhood). There exists an open neighborhood N_ϵ that is diffeomorphic to the total space of the normal bundle over M in A.
  - This gives $H^*(A, A - M) \simeq H^*(N_\epsilon, N_\epsilon - M) \simeq H^*(E, E_0)$.
  - Consider the composition $H^k(A, A - M) \to H^k(A) \to H^k(M)$. The image of the fundamental class u' in $\mathbb{Z}/2\mathbb{Z}$ coefficient (or $\mathbb{Z}$ coefficient if ν_M is orientable) is $w_k(\nu^k)$ (or $e(\nu_k)$ if orientable).
  - The image of u' in $H^k(A)$ is called the dual cohomology class of M in the sense of Poincaré duality. For instance, if $M^n \subset \mathbb{R}^{n+k}$, then $w_k(\nu_M)$ or $e(\nu_M) = 0$ as the dual cohomology class in $H^k(\mathbb{R}^{n+k})$ is zero.
- Note that M is orientable iff TM is orientable: $H^n(M, M - x) \simeq H^n(T_xM, T_xM - 0)$ via the results above. We can now push the generator that defines the orientation from one to the other by local arguments.
- (Slant Product). Consider $H^*(X) \otimes H^*(Y) \otimes H_*(Y) \to H^*(X)$. Now replace $H^*(X) \otimes H^*(Y)$ by $H^*(X \times Y)$ gives $H^*(X \times Y) \otimes H_*(Y) \to H^*(X)$, where the image of $p \otimes \beta$ is denoted as p/β.
  - Let the image of $u' \in H^n(M \times M, M \times M - M)$ to be $u'' \in H^n(M \times M)$. $u''/\mu = 1 \in H^0(M)$.

- Assume M orientable (or coefficient $\mathbb{Z}/2\mathbb{Z}$). We can define dual basis $(b_j^{\#})$ for each basis (b_j) such that $\langle b_i \cup b_j^{\#}, \mu\rangle = \delta_{ij}$. This gives an identification $H^k(M) \simeq H^{n-k}(M)$. (Note that such an isomorphism is not canonical when M is not orientable.) Computation gives that $u'' = \sum(-1)^{\dim b_i} b_i \times b_i^{\#}$. This tells $e(TM) = \sum(-1)^{\dim b_i} b_i \cup b_i^{\#}$, which tells that $\langle e(TM), \mu\rangle = \sum(-1)^{\dim b_i} = \chi(M)$. (Or replace everything by $w_n(TM)$ and modulo 2. Note that $TM \simeq \nu_M$ where the latter is its normal bundle in $M \times M$).
- (Wu Class). Consider the homomorphism sending $x \in H^{n-k}(M)$ to $\langle \mathrm{Sq}^k(x), \mu\rangle \in \mathbb{Z}/2\mathbb{Z}$. This is represented by $v_k \in H^k(M)$. $v = \sum v_i$ is the total Wu class, which satisfies $\langle v \cup x, \mu\rangle = \langle \mathrm{Sq}(x), \mu\rangle$ where $\mathrm{Sq} = \sum \mathrm{Sq}^i$. Now $w_k = \sum_{i+j=k} \mathrm{Sq}^i(v_j)$.

7.4 Local Systems and Cohomology

- A local system on X is a locally constant sheaf. If X path-connected, then equivalently, a local system with constant stalk L corresponds to a group homomorphism $\rho : \pi_1(X, x) \to \mathrm{Aut}(L)$.
  - On one hand, let γ be a loop at x. It is easy to show that local systems over $[0, 1]$ is constant, so in particular is $\gamma^*\mathcal{L}$. This gives an isomorphism $(\gamma^*\mathcal{L})_0 \simeq (\gamma^*\mathcal{L})_1$ via parallel transport, i.e., an automorphism of L. On the other hand, let ρ be such a homomorphism. We consider the universal covering $\tilde{X}$ over X and define the local system via $\mathcal{L}(U) = \{f : \tilde{U} \to L, f(\gamma y) = \rho(\gamma)f(y)\}$ where $y \in \tilde{U}$ and $\gamma \in \pi_1(X)$.
  - We can define $H^j(X; \mathcal{L})$ via sheaf cohomology.

7.5 Obstructions

- Consider the fiber bundle $V_k(\xi)$ with fibers $V_k(F)$. There exists a cross-section over the $(n-k)$-skeleton of B, and a cross-section over the $(n-k+1)$-skeleton exists iff certain class $\mathfrak{o}_j(\xi) \in H^j(B; \{\pi_{j-1}V_{n-j+1}(F)\})$ where $\{\pi_{j-1}V_{n-j+1}(F)\}$ is the local coefficient system where over each x there is $\pi_{j-1}V_{n-j+1}(F_x)$ where $j = n-k+1$. The reduction of $\mathfrak{o}_j(\xi)$ to $\mathbb{Z}/2\mathbb{Z}$-coefficient gives $w_j(\xi)$.
  - In fact the obstructions $\mathfrak{o}_j(\xi)$ are completely determined by the Stiefel-Whitney classes.
  - If ξ oriented, then $\mathfrak{o}_n(\xi) = e(\xi)$.
- (Gysin). Suppose ξ oriented. There is a LES:

$$... \longrightarrow H^k(B) \xrightarrow{\cup e} H^{n+k}(B) \longrightarrow H^{n+k}(E_0) \longrightarrow H^{k+1}(B) \longrightarrow ...$$

  This comes from the LES of cohomology group via the sequence $E_0 \to E \to (E, E_0)$ and the identification $H^*(B) \simeq H^*(E)$ and $H^{*+n}(E, E_0) \simeq H^*(B)$. The analogous holds for unoriented ξ.
  - In particular consider the 2-fold cover $\tilde{B}$ and the line bundle over B with total space $\tilde{B} \times \mathbb{R}/((x, t) \sim (x', -t))$. We get a LES

$$... \longrightarrow H^k(B) \xrightarrow{\cup w_1} H^{k+1}(B) \longrightarrow H^{k+1}(\tilde{B}) \longrightarrow H^{k+1}(B) \longrightarrow ...$$

- The 2-fold cover $\tilde{\mathrm{Gr}}_n$ and the universal bundle $\tilde{\gamma_n}$ classifies oriented n-bundles. $H^*(\tilde{\mathrm{Gr}}_n; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[w_2, ..., w_n]$ via the sequence above.

7.6 Complex Vector Bundles and Chern Class

- A complex structure on a $2n$-dimensional v.b. can be viewed as a $\mathbb{R}$-linear map $J : E(\xi) \to E(\xi)$ such that $J(J(v)) = -v$ for each v. This in particular defines the action of $\mathbb{C}$ on $E(\xi)$ via $i \cdot v = J(v)$. Let ω be a complex bundle, we denote the underlying real bundle as $\omega_{\mathbb{R}}$.

- A complex function $f : U \to U'$ where $U \subset \mathbb{C}^n$ and $U' \subset \mathbb{C}^m$ is holomorphic if it is holomorphic coordinate-wise. On the other hand, it is holomorphic iff the induced maps on tangent spaces are complex linear. Implicitly, being complex linear is equivalent to satisfying the Cauchy-Riemann equations.

- A complex manifold is a manifold with a complex structure J on TM, such that locally it is diffeomorphic to an open subset of $\mathbb{C}^n$ such that the derivative is complex linear everywhere ($dh \circ J = J_0 \circ dh$). A map between two complex manifolds is holomorphic if Df is complex linear everywhere.

- Note that a complex vector bundle automatically has an orientation (by taking a basis $(v_1, iv_1, ..., v_n, iv_n)$ for a basis $(v_1, ..., v_n)$), since a complex linear transformation preserves real orientation. We can therefore talk about $e(\omega_{\mathbb{R}}) \in H^{2n}(B; \mathbb{Z})$.

- A Hermitian metric is an Euclidean metric such that $|iv| = |v|$, and this defines an inner product such that $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

- (Chern Class). We define a $(n-1)$-complex bundle ω_0 over E_0 by letting the fiber over (x, v) to be the orthogonal complement of v in E_x. Now, recall that Gysin sequence gives $... \to H^{i-2n}(B) \to H^i(B) \to H^i(E_0) \to H^{i+1-2n}(B) \to ...$, so in particular $\pi_0^* : H^i(B) \simeq H^i(E_0)$ if $i < 2n-1$. We now define $c_n(\omega) := e(\omega)$ and $c_i(\omega) := (\pi_0^*)^{-1} c_i(\omega_0)$. For $i > n$ $c_i(\omega) := 0$.
  - It is easily shown that the Chern classes are natural and invariant under $\oplus\epsilon^k$.
  - To compute the cohomology of $\mathbb{CP}^k$, note that we have Gysin sequence:

    $$... \longrightarrow H^i(E_0) \longrightarrow H^i(\mathbb{CP}^k) \xrightarrow{\cup c_1} H^{i+2}(\mathbb{CP}^k) \longrightarrow H^{i+1}(E_0) \longrightarrow ...$$

    Since $E_0 \simeq S^{2k+1}$, we have that $H^i(\mathbb{CP}^k) \simeq H^{i+2}(\mathbb{CP}^k)$ which is 0 if i odd and $\mathbb{Z}$ (generated by $c_1^{i/2}$) if i even. In particular, $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[c_1]/(c_1^{n+1})$. Let $n \to \infty$ gives $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1]$.
  - In fact, $H^*(\mathrm{Gr}_n) = \mathbb{Z}[c_1, ..., c_n]$. This is proved via induction and Gysin sequence. As expected, Gr_n classifies rank n complex bundles, in a unique way up to homotopy.
  - $c(\omega \oplus \eta) = c(\omega)c(\eta)$, just as in the real case.

- Let ω be a complex bundle, we can define its conjugate bundle $\bar{\omega}$ via the opposite complex structure. It is not guaranteed that $\omega \simeq \bar{\omega}$. $c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - ... \pm c_n(\omega)$, since the underlying real orientation $(v_1, iv_1, ..., v_n, iv_n)$ and $(v_1, -iv_1, ..., v_n, -iv_n)$ differ by a sign $(-1)^n$.
  - We can also define the dual bundle $\mathrm{Hom}(\omega, \mathbb{C})$, and if ω admits a Hermitian metric, then $\mathrm{Hom}(\omega, \mathbb{C}) \simeq \bar{\omega}$ via sending v to $\langle -, v\rangle$.
  - For example, $c(\tau^n) = (1 + a)^{n+1}$ where τ^n is the tangent bundle over $\mathbb{CP}^n$ and $a = -c_1(\gamma^1)$: Let ω^n be the orthogonal complement of γ^1 over $\mathbb{CP}^{n+1}$, we have $\tau^n \simeq \mathrm{Hom}(\gamma^1, \omega^n)$, and adding ϵ^1 on both sides gives $\tau^n \oplus \epsilon^1 \simeq \mathrm{Hom}(\gamma^1, (\epsilon^1)^{n+1})$, so $c(\tau^n) = c(\mathrm{Hom}(\gamma^1, \epsilon^1)^{n+1}) = (1 - c_1(\gamma^1))^{n+1}$.

7.7 Pontryagin Class

- The complex structure on $\xi \otimes \mathbb{C}$ is via $J(x, y) = (-y, x)$. $\xi \otimes \mathbb{C}$ is canonically isomorphic to $\overline{\xi \otimes \mathbb{C}}$, with the correspondence $(x, y) \to (x, -y)$. In particular, $c_i(\xi \otimes \mathbb{C})$ is of order 2 for odd i.

- (Pontryagin Class) The ith Pontryagin class is defined to be $p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Z})$. $p_i(\xi) = 0$ for $i > n/2$.
  - As for Chern class, $p_i(\xi) = p_i(\xi \oplus \epsilon)$.

– $p(\xi \oplus \eta)$ is congruent to $p(\xi)p(\eta)$ modulo elements of order 2. In other words, $2(p(\xi \oplus \eta) - p(\xi)p(\eta)) = 0$.

- For a complex bundle ω, we have a natural isomorphism $(\omega_{\mathbb{R}}) \otimes \mathbb{C} \simeq \omega \oplus \bar{\omega}$. In particular, we can compute $c(\omega_{\mathbb{R}} \otimes \mathbb{C}) = c(\omega)c(\bar{\omega})$ and hence get a formula for the Pontryagin classes.

- For an oriented bundle ξ, we have an isomorphism $(\xi \otimes \mathbb{C})_{\mathbb{R}} \simeq \xi \oplus \xi$, which either preserves or reverses the orientation depending on if $n(n-1)/2$ is even or odd. Indeed, the canonical basis for the LHS is $(v_1, iv_1, ..., v_n, iv_n)$ and for the RHS is $(v_1, ..., v_n, iv_1, ..., iv_n)$. In particular, this gives $p_k(\xi) = e(\xi)^2$ if ξ is of rank $2k$.

- (Oriented Grassmanian) $\tilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$ is the collection of n-dimensional oriented subspaces of $\mathbb{R}^\infty$. It can be viewed as $BSO(n)$.

  – Let Λ be an ID containing $\frac{1}{2}$. $H^*(\tilde{\mathrm{Gr}}_{2m+1})$ is a polynomial ring over Λ generated by $p_1(\tilde{\gamma}^{2m+1}), ..., p_m(\tilde{\gamma}^{2m+1})$. And $H^*(\tilde{\mathrm{Gr}}_{2m})$ is a polynomial ring over Λ generated by $p_1(\tilde{\gamma}^{2m}), ..., p_{m-1}(\tilde{\gamma}^{2m})$ and $e(\tilde{\gamma}^{2m})$.

- (Chern Numbers). Let K^n compact complex manifold. Let $I = (i_1, ..., i_r)$ a partition of n, then the I-th Chern number is $c_I(K^n) = c_{i_1}...c_{i_r}[K^n] = \langle c_{i_1}...c_{i_r}, \mu_{2n}\rangle$ where μ_{2n} is the fundamental class that gives the orientation.

  – Note that $H^{2n}(\mathrm{Gr}_n)$ has basis $c_{i_1}...c_{i_r}$ where $(i_1, ..., i_r)$ gives a partition. Also, the tangent bundle over K^n is classified by a map $K^n \to \mathrm{Gr}_n$. Now, the Chern numbers are exactly $\langle c_{i_1}(\gamma^n)...c_{i_r}(\gamma^n), f_*(\mu_{2n})\rangle$ which gives exactly what $f_*(\mu_{2n})$ is.

- (Pontryagin Numbers). We similarly define $p_I[M^{4n}] = \langle p_{i_1}...p_{i_r}, \mu_{4n}\rangle$ where μ_{4n} is the fundamental class on M^{4n}.

  – Note that the orientation of a complexified bundle does not depend on the original orientation, so Pontryagin numbers are negated if the orientation changes. In particular, if some Pontryagin numbers are non-zero, then M does not admit an orientation-reversing diffeomorphism.

  – In particular, as the case for Stiefel-Whitney numbers, if M^{4n} has some non-zero Pontryagin numbers, then it cannot be the boundary of some W^{4n+1}.

- Similar to the real case, $H^*(\mathrm{Gr}_n; \mathbb{Z})$ is mapped isomorphically to the ring of symmetric functions in $\mathbb{Z}[a_1, ..., a_n]$. Let s_I to be the polynomial such that $s_I(\sigma_1, ..., \sigma_k) = \sum x_1^{i_1}...x_r^{i_r}$ where $I = (i_1, ..., i_r)$ is a partition of k, then $(s_I(c_1, ..., c_k))_I$ form a basis for $H^{2k}(\mathrm{Gr}_n; \mathbb{Z})$.

- Denote $s_I(c)[K^n] = s_I[K^n] = \langle s_I(c(\tau^n)), \mu_{2n}\rangle \in \mathbb{Z}$. Let $K^1, ...K^n$ be complex manifolds with $s_k[K^k] \neq 0$. The matrix $[c_I[K^{j_1}...K^{j_r}]]$ where I and $J = (j_1, ..., j_s)$ ranges over all the partitions of n is non-singular. Similar result holds for the Pontryagin numbers.

7.8 Cobordism

- M and M' lie in the same oriented cobordism class if there exists smooth compact oriented X such that $\partial X \simeq M + (-M')$. Let Ω_n be the Abelian group of all oriented cobordism classes. We have naturally $\Omega_m \times \Omega_n \to \Omega_{m+n}$. This gives Ω_* a graded ring structure and is commutative in the graded sense.

  – For each partition of k, $M^{4k} \to p_I[M]$ gives a group homomorphism $\Omega_{4k} \to \mathbb{Z}$. We can show $\mathrm{Rank}\,\Omega_{4k} \geq p(k)$ by some computations with the Pontryagin numbers ($\mathbb{CP}^{2i_1} \times ... \times \mathbb{CP}^{2i_r}$ are linearly independent in Ω_{4k}). In fact, we can show $\mathrm{Rank}\,\Omega_{4k} = p(k)$.

- (Thom Space). Assume ξ admits an Euclidean metric. $T(\xi) = E(\xi)/A$, where A is the subspace consisting of all points (x, v) where $|v| \geq 1$. $T(\xi)$ has a canonical basepoint, t_0, so that $T(\xi) - t_0 = \{(x, v), |v| < 1\}$.

- Assume B is a CW complex, then $T(\xi)$ is a $(k-1)$ connected CW complex, with one $(n+k)$-cell corresponding to each n-cell of B.
- Assume ξ is oriented (so Thom isomorphism is effective), then $H_{n+k}(T(\xi), t_0) \simeq H_{n+k}(T(\xi), T_0) \simeq H_{n+k}(E, E_0) \simeq H_n(B)$.

- A map between Abelian groups is an $\mathcal{E}$-isomorphism if the kernel and cokernel are both finite. In particular, suppose ξ^k is oriented over a finite complex, then $\pi_{n+k}(T) \to H_{n+k}(T) \simeq H_n(B)$ is an $\mathcal{E}$-isomorphism.

- Recall that a regular value y of a smooth map $f : M \to N$ is $y \in N$ such that $T_xM \to T_yN$ is surjective for each $x \in f^{-1}(y)$. $f^{-1}(y)$ is a $(m-n)$-submanifold if y is regular and the set of regular values is dense.
  - Note that y is a regular value iff f is transverse to y. Let $Y \subset N$ be an $(n-k)$-submanifold, then f is transverse to Y if $T_xM + T_yY \to T_yN$ is surjective, and if so then $f^{-1}(Y)$ is a $(m-k)$-submanifold. Normal bundles of Y and $f^{-1}(Y)$ can be identified.

- Every continuous $f : S^m \to T(\xi)$ is homotopic to some g which is smooth on $g^{-1}(T - t_0)$, and is transverse to the zero-section B. The oriented cobordism class of $g^{-1}(B)$ depends only on the homotopy type of g. In particular, there is a map $\pi_m(T(\xi), t_0) \to \Omega_{m-k}$.
  - (Thom). For $k > n+1$, $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \simeq \Omega_n$ canonically. Similarly, $\pi_{n+k}(T(\gamma^k), t_0) \simeq \mathcal{N}_n$ canonically, where $\mathcal{N}_n$ is the unoriented cobordism group.
  - Ω_n is finite if $4 \nmid n$, and is finitely generated of rank $p(k)$ if $n = 4k$. In particular, $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ generated by $(\mathbb{CP}^{2i})$.

7.9 Multiplicative Sequences

- (Multiplicative Sequences). Let A^* be a graded Λ-algebra, and A^Π be the ring of formal series in A. A multiplicative sequence is defined as a sequence of polynomials $K_1(x_1), K_2(x_1, x_2), \ldots$ such that each K_n is homogeneous of degree n, and that if $K(a) = 1 + K_1(a_1) + \ldots$, then $K(ab) = K(a)K(b)$. We give some examples:
  - Recall $c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \ldots \pm (-1)^n c_n(\omega)$. The polynomials $K_n(x_1, \ldots, x_n) = (-1)^n x_n$ gives a multiplicative sequence for this.
  - Recall that if $\eta \oplus \omega = 1$, then $p(\eta)p(\omega) = 1$ (or $c(\eta)c(\omega)$, or $(w(\eta)w(\omega))$. The polynomials $K_n = \sum_{i_1+2i_2+\ldots+ni_n=n} \frac{(i_1+\ldots i_n)!}{i_1!\ldots i_n!}(-x_1)^{i_1}\ldots(-x_n)^{i_n}$ characterizes such $(K(a) = a^{-1})$.
  - Recall that we have $1 - p_1 + p_2 - \ldots \pm p_n = c(\omega)c(\bar{\omega})$. This is characterized by $K_{2n+1} = 0$ and $K_{2n} = x_n^2 - 2x_{n-1}x_{n+1} + \ldots \mp 2x_1x_{2n-1} \pm 2x_n$.'
  - (Hirzeburch). Let $A^* = \Lambda[t]$. For a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \ldots$, there exists a unique multiplicative sequence (K_n) such that $K(1+t) = f(t)$.

- Let K be a multiplicative sequence in $\mathbb{Q}$-coefficients. For a compact oriented smooth manifold M^m, we define $K([M]) = 0$ if $4 \nmid m$ and $K([M]) = \langle K_n(p_1, \ldots, p_n), \mu_{4n} \rangle$ if $4 \mid m$. This gives a ring homomorphism $\Omega_* \to \mathbb{Q}$.

- (Signature). The signature for a compact oriented smooth manifold M^m is 0 if $4 \nmid m$ and the number of positive entries on the diagonal matrix (after selecting the correct basis (a_i) of $H^{2n}(M^{4n}; \mathbb{Q})$) on $[\langle a_i \cup a_j, \mu \rangle]$ if $4 \mid m$.
  - $\sigma(M + M') = \sigma(M) + \sigma(M')$, $\sigma(M \times M') = \sigma(M)\sigma(M')$. If $M = \partial W$, then $\sigma(M) = 0$.
  - (Signature Theorem). Let (L_k) be the multiplicative sequence belonging to the power series $\sqrt{t}/\tanh\sqrt{t}$. Then $\sigma(M^{4n}) = L([M])$. This only depends on the oriented homotopy type of M.

- Given a multiplicative sequence K with coefficient in Λ, an ID with 2 invertible, we can define $k_n(\xi) = K_n(p_1(\xi), ..., p_n(\xi)) \in H^{4n}(B; \Lambda)$. This satisfies naturality and that $k(\eta \oplus \omega) = k(\eta)k(\omega)$. On the other hand, multiplicative characteristic classes k_n satisfying such properties come from some unique multiplicative sequence K.